

Exceptional Moufang quadrangles and structurable algebras

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Abstract

In 2000, J. Tits and R. Weiss classified all Moufang spherical buildings of rank two, also known as Moufang polygons. The hardest case in the classification consists of the Moufang quadrangles. They fall into different families, each of which can be described by an appropriate algebraic structure. For the exceptional quadrangles, this description is intricate and involves many different maps that are defined *ad hoc* and lack a proper explanation.

In this paper, we relate these algebraic structures to two other classes of algebraic structures that had already been studied before, namely to Freudenthal triple systems and to structurable algebras. We show that these structures give new insight in the understanding of the corresponding Moufang quadrangles.

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1 Introduction

In 1974, Jacques Tits published his famous lecture notes “Buildings of Spherical Type and Finite BN-Pairs” [T74], in which he classified all spherical buildings of rank at least 3. The main motivation for studying buildings is given by the following quote, which we have taken from [T74, p. V].

The origin of the notions of buildings and BN-pairs lies in an attempt to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups and, in particular, the exceptional groups.

It took another 26 years before the analogous result in lower rank, namely the classification of spherical buildings of rank 2 satisfying the Moufang property, was finished by Jacques Tits and Richard Weiss [TW]. There is no doubt that the hardest part in the whole classification is precisely where the exceptional quadrangles turn up, and in fact, there is an ongoing effort to try to understand the structure of these exceptional quadrangles and the corresponding rank 2 forms of the exceptional linear algebraic groups.

A first attempt to provide an algebraic framework to describe the Moufang quadrangles was given by the second author [D], who introduced *quadrangular systems*, an algebraic structure consisting of a pair of groups intertwined in a very delicate way, with no less than 20 defining axioms. Unfortunately, this algebraic structure is not very well suited for getting a deeper algebraic understanding of the specific examples, in particular the exceptional ones; its main purpose is to have a uniform algebraic structure to describe all possible Moufang quadrangles.

A second attempt, which is more focused on the exceptional Moufang quadrangles, was provided by Richard Weiss [W1], who developed a theory of *quadrangular algebras*. They describe the algebraic structures needed to construct the exceptional Moufang quadrangles as a generalization of pseudo-quadratic spaces.

Inspired by discussions that we had with Skip Garibaldi, we became aware of the existence of a large class of non-associative algebras called *structurable algebras*, which have been introduced by Bruce Allison in 1978 [A2] in the context of exceptional Lie algebras. These structurable algebras were known to be related to yet another class of algebraic structures, known as *Freudenthal triple systems*, introduced by Kurt Meyberg in 1968 [Me]. We immediately point out one drawback of both algebraic systems: at the moment, they are only defined over fields of characteristic different from 2, and very often (but this seems less essential) also different from 3.

It turns out that every quadrangular algebra over a field of good characteristic can be made into a Freudenthal triple system, and consequently also into a structurable algebra; see Theorem 5.1 below. Even in the case of the exceptional quadrangular algebras (i.e. those of type E_6 , E_7 and E_8) these structurable algebras are well understood and can be nicely described; see Theorem 7.1 below. It is our hope that these new insights will lead to a better understanding of the corresponding Moufang quadrangles.

We point out that the algebraic structures that we obtain, are essentially describing the structure of the (non-abelian) rank one residues; nevertheless, they are related with the module structure (arising from the rank two structure) in a surprisingly simple fashion (see Theorem 4.3 and Corollary 5.8 below).

Organization of the paper

The paper is organized as follows. In section 2, we first recall the notion of a Moufang polygon, and we explain how many of the examples are associated to linear algebraic groups of relative rank two. The reader can safely skip this section if he wishes; it is not directly used in the rest of the paper, but explains why the study of quadrangular algebras is of interest. In section 3, we recall some known facts about the different types of algebraic structures that we will encounter, namely quadrangular algebras, Freudenthal triple systems, and structurable algebras.

In section 4, we explain the connection between quadrangular algebras and Freudenthal triple systems. In the main section 5, we explain how every quadrangular algebra can be made into a structurable algebra; we have to use some explicit form of Galois descent in order to arrive at an exact description of those algebras. We then have a closer look at the case of pseudo-quadratic quadrangular algebras in section 6, and at the case of quadrangular algebras of type E_6 , E_7 and E_8 in the final section 7.

Acknowledgments

We thank Richard Weiss for several very interesting and stimulating discussions on the exceptional Moufang quadrangles, and in particular we were inspired by an unpublished manuscript of him that we could use in the proof of Lemma 7.5. We are grateful to Detlev Hoffmann for pointing out to us why the Arason invariant determines the quadratic forms of type E_8 up to similarity (see Remark 7.7). We also express our gratitude to Skip Garibaldi for many fruitful discussions, and in particular for making us aware of the theory of structurable algebras, which was completely unknown to us.

Finally, we are extremely thankful to the referee who did an absolutely amazing job by carefully reading an earlier version of the paper from A to Z, and suggesting many relevant improvements.

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2 Moufang polygons

A *Moufang polygon* is a notion from incidence geometry introduced by Jacques Tits. We only give a brief summary of the theory of Moufang polygons, and we refer to [TW] or [DV] for more details. The importance will immediately become clear in Theorem 2.1 below.

2.1 Definitions

A *generalized n -gon* Γ is a connected bipartite graph with diameter n and girth $2n$, where $n \geq 2$. If we do not want to specify the value of n , then we call this a *generalized polygon*. We call a generalized polygon *thick* if every vertex has at least 3 neighbors. A *root* in Γ is a (non-stammering) path of length n in Γ . Observe that the two extremal vertices of such a path are always *opposite*, i.e. their distance is equal to the diameter n of Γ . An *apartment* in Γ is a circuit of length $2n$.

Let Γ be a thick generalized n -gon with $n \geq 3$, and let $\alpha = (x_0, \dots, x_n)$ be a root of Γ . Then the group U_α of all automorphisms of Γ fixing all neighbors of x_1, \dots, x_{n-1} (called a *root group*) acts freely on the set of vertices incident with x_0 but different from x_1 . If U_α acts transitively on this set (and hence regularly), then we say that α is a *Moufang root*. It turns out that α is a Moufang root if and only if U_α acts regularly on the set of apartments through α .

A *Moufang polygon* is a generalized n -gon for which every root is Moufang. We then also say that Γ satisfies the *Moufang condition*. The group generated by all the root groups is sometimes called the *little projective group* of Γ .

Moufang polygons have been classified by J. Tits and R. Weiss [TW]. Loosely speaking, the result is the following.

Theorem 2.1 ([TW]). *Let Γ be a Moufang n -gon. Then $n \in \{3, 4, 6, 8\}$. Moreover, every Moufang polygon arises from an absolutely simple linear algebraic group of relative rank 2, or from a corresponding classical group or group of mixed type.*

In particular, every Moufang polygon is of “algebraic origin”, and in fact, the Moufang polygons provide a useful tool to help in the understanding of the corresponding groups; this is particularly true for the Moufang polygons arising from linear algebraic groups of exceptional type. For instance, the Kneser–Tits problem for groups of type $E_{8,2}^{66}$ has recently been solved using the theory of Moufang polygons [PTW].

In order to describe a Moufang polygon in terms of algebraic data, we will use so-called *root group sequences*. A root group sequence for a Moufang n -gon is a sequence of n root groups, labeled U_1, \dots, U_n , together with *commutator relations* describing how elements of two different root groups U_i and U_j commute. In each case, the commutator of an element of U_i and U_j (with $i < j$) belongs to the group $\langle U_{i+1}, \dots, U_{j-1} \rangle$. The following result is crucial.

Theorem 2.2. *Let Γ be a Moufang n -gon. Then Γ is completely determined by the root groups U_1, \dots, U_n together with their commutator relations.*

Proof. See [TW, Chapter 7]. □

For more details about this procedure, and how the Moufang polygons can be reconstructed from the root group sequences, we refer to [TW] or to the survey article [DV].

For each type of Moufang polygons, we will describe an *algebraic structure* which will allow us to parametrize the root groups and describe the commutator relations.

2.2 Algebraic structures for Moufang polygons

We give an overview of the classification of Moufang polygons and of the algebraic structures involved in this classification. This section contains more information than we will actually need, but it puts our theory in a broader context, which is not so easy to find in the existing literature.

2.2.1 Moufang triangles

Every Moufang triangle (i.e. a Moufang projective plane) can be described in terms of an *alternative division algebra*, i.e. a division algebra A which is

not necessarily associative, but which instead satisfies the weaker identities

$$a^{-1}(ab) = b = (ba)a^{-1} \quad \text{for all } a, b \in A \setminus \{0\}.$$

These algebras have been classified by Bruck and Kleinfeld: either they are associative after all, or they are 8-dimensional *Cayley–Dickson division algebras*, also known as *octonion algebras*.

If A is such an alternative division algebra, then we define $U_1 \cong U_2 \cong U_3 \cong (A, +)$; we denote the explicit isomorphisms from A to U_i by x_i , and we call this the *parametrization* of the groups U_i by $(A, +)$. The commutator relations are then given by

$$[x_1(a), x_3(b)] = x_2(ab)$$

for all $a, b \in A$; note that the other commutators $[U_1, U_2]$ and $[U_2, U_3]$ are trivial. Every Moufang triangle can be described in this fashion.

If A is a finite-dimensional division algebra of degree d , then this Moufang triangle arises from a linear algebraic group of absolute type A_{3d-1} .

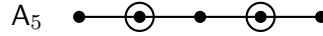


Figure 1: Moufang triangle parametrized by a quaternion division algebra

If A is infinite-dimensional, then the associated group is no longer an algebraic group, but it can still be viewed as a classical group, namely $\mathrm{PSL}_3(A)$.

The case where A is an octonion division algebra is exceptional, and arises from a linear algebraic group of absolute type E_6 .

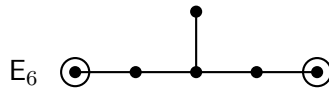


Figure 2: Moufang triangle parametrized by an octonion division algebra

2.2.2 Moufang hexagons

Every Moufang hexagon can be described in terms of a (*quadratic*) *Jordan division algebra of degree 3*, or equivalently, by *anisotropic cubic norm structures*. We will not give a precise definition of these algebraic structures since we will not need them explicitly, but we refer to [DV, KMRT, TW] instead.

We will only mention that if J is such an anisotropic cubic norm structure over a field K , then either J/K is a purely inseparable cubic extension, or $\dim_K J \in \{1, 3, 9, 27\}$.

If J is such an anisotropic cubic norm structure, then we define $U_1 \cong U_3 \cong U_5 \cong (J, +)$ and $U_2 \cong U_4 \cong U_6 \cong (K, +)$. The commutator relations can be expressed in terms of the norm, trace and Freudenthal cross product, but their explicit form is not important for us; we refer to [DV, TW] for more details.

The case where $J = K$ gives rise to the so-called *split Cayley hexagon*, which arises from a split linear algebraic group of type G_2 .



Figure 3: Moufang hexagon parametrized by a commutative field

If J is a cubic separable extension field of K , then the corresponding Moufang hexagon is the so-called *twisted triality hexagon*, which arises from a quasi-split linear algebraic group of type 3D_4 or 6D_4 .

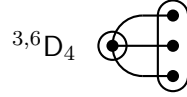


Figure 4: Moufang hexagon parametrized by a cubic extension

If J is a purely inseparable cubic extension field of K , then the corresponding Moufang hexagon arises from a so-called group of mixed type; such a group is a subgroup (as an abstract group) of an algebraic group of type G_2 , but it is defined over the *pair of fields* (K, J) instead of over a single field. See, for instance, [T74] for more information on these groups of mixed type.

The next case is where J is a 9-dimensional K -algebra. There are two cases to distinguish; either J is a central simple cubic cyclic division algebra, or it is a twisted form of such an algebra, arising from an involution of the second kind on such an algebra. The resulting Moufang hexagons arise from linear algebraic groups of type E_6 and 2E_6 , respectively.

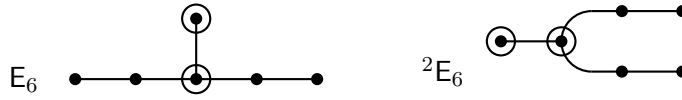


Figure 5: Moufang hexagons parametrized by 9-dimensional cubic norm structures

Finally, if J is 27-dimensional over K , then it is an Albert division algebra, and the resulting Moufang hexagons arise from linear algebraic groups of absolute type E_8 .

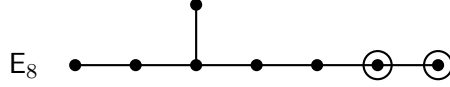


Figure 6: Moufang hexagons parametrized by Albert division algebras

2.2.3 Moufang octagons

The Moufang octagons have a fairly simple structure from an algebraic point of view. Every Moufang octagon can be described from a commutative field K with $\text{char}(K) = 2$ equipped with a *Tits endomorphism* σ , i.e. an endomorphism such that $(x^\sigma)^\sigma = x^2$ for all $x \in K$. The root groups U_1, U_3, U_5, U_7 are parametrized by $(K, +)$, and the root groups U_2, U_4, U_6, U_8 are parametrized by some non-abelian group T with underlying set $K \times K$, and with group operation

$$(a, b) \cdot (c, d) := (a + c, b + d + a^\sigma c) \quad \text{for all } a, b, c, d \in K.$$

We do not go into more details, and we again refer to [TW]. The corresponding groups are the Ree groups of type 2F_4 .

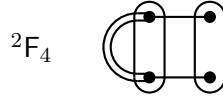


Figure 7: Moufang octagons

2.2.4 Moufang quadrangles

We finally come to the most involved case in the classification, which is the case of the Moufang quadrangles. In principle, it is possible to define a single algebraic structure to describe all possible Moufang quadrangles; this gives rise to the so-called *quadrangular systems* which have been introduced by the second author [D]. These structures, however, have some disadvantages from an algebraic point of view; most notably, the definition does not mention an underlying field of definition (although it is possible to construct such a field from the data), and the axiom system looks very wild and complicated, with no less than 20 defining identities.

Instead, we will follow the original classification as given by Tits and Weiss in [TW], distinguishing six different (non-disjoint) classes:

- (1) Moufang quadrangles of indifferent type;
- (2) Moufang quadrangles of quadratic form type;
- (3) Moufang quadrangles of involutory type;

- (4) Moufang quadrangles of pseudo-quadratic form type;
- (5) Moufang quadrangles of type E_6 , E_7 and E_8 ;
- (6) Moufang quadrangles of type F_4 .

The Moufang quadrangles of types (2)–(4) are often called *classical*, those of type (5) and (6) are called *exceptional* and those of type (1) are of *mixed type*. Since the Moufang quadrangles of type (1) and (6) only exist over fields of characteristic two, and moreover are not directly related to rank two forms of algebraic groups, we will exclude those two classes from our further discussion.

Moufang quadrangles of quadratic form type Moufang quadrangles of quadratic form type are determined by an anisotropic quadratic form $q: V \rightarrow K$, where V is an arbitrary (possibly infinite-dimensional) vector space over some commutative field K . The root groups U_1 and U_3 are parametrized by $(K, +)$ and the root groups U_2 and U_4 are parametrized by $(V, +)$; the commutator relations will involve the quadratic form q and its corresponding bilinear form f . If $d = \dim_K V$ is finite, then these Moufang quadrangles arise from algebraic groups; they are of absolute type $B_{\ell+2}$ if $d = 2\ell + 1$ is odd, and of type $D_{\ell+2}$ if $d = 2\ell$ is even.

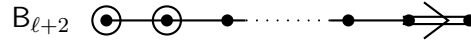


Figure 8: Moufang quadrangles parametrized by quadratic forms (odd dimension)



Figure 9: Moufang quadrangles parametrized by quadratic forms (even dimension)

Moufang quadrangles of involutory type Moufang quadrangles of involutory type are determined by a (skew) field K equipped¹ with an involution σ . The root groups U_2 and U_4 are parametrized by $(K, +)$ and the root groups U_1 and U_3 are parametrized by $(\text{Fix}_K(\sigma), +)$. If K is finite-dimensional over its center, of degree d , then these Moufang quadrangles arise from algebraic groups; they are of absolute type ${}^2A_{4d-1}$ if the involution is of the second kind, and they are of absolute type ${}^1D_{2d}$ if the involution is of the first kind.

¹If $\text{char}(K) = 2$, some more data are required, but this is not relevant for our purposes.

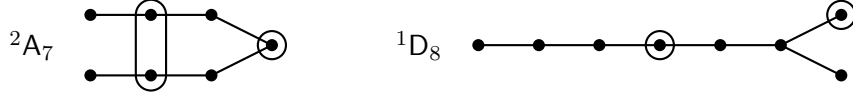


Figure 10: Some Moufang quadrangles of involutory type

Moufang quadrangles of pseudo-quadratic form type Moufang quadrangles of pseudo-quadratic form type are determined² by an anisotropic pseudo-quadratic space.

Definition 2.3 ([W1, Definition 1.16]). A pseudo-quadratic space is a quintuple (L, σ, X, h, π) where

- (i) L is a skew field;
- (ii) σ is an involution of L , and we let $K := \text{Fix}_L(\sigma)$;
- (iii) X is a right vector space over L ;
- (iv) $h: X \times X \rightarrow L$ is a *skew-hermitian form*, i.e.
 - h is bi-additive and $h(x, yu) = h(x, y)u$, and
 - $h(x, y)^\sigma = -h(y, x)$,
for all $x, y \in X$ and all $u \in L$;
- (v) π is a *pseudo-quadratic form* from X to L , i.e.
 - $\pi(x + y) \equiv \pi(x) + \pi(y) + h(x, y) \pmod{K}$, and
 - $\pi(xu) \equiv u^\sigma \pi(x)u \pmod{K}$,
for all $x, y \in X$ and all $u \in L$.

A pseudo-quadratic space (L, σ, X, h, π) is called *anisotropic* if

$$\pi(x) \equiv 0 \pmod{K} \text{ only if } x = 0.$$

Remark 2.4. If $\text{char}(L) \neq 2$, then the pseudo-quadratic form π is completely determined (modulo K) by the skew-hermitian form h . We have nevertheless decided to include the pseudo-quadratic form in the definition, because this form will play an important role in the sequel.

The root groups U_2 and U_4 are parametrized by $(L, +)$; the root groups U_1 and U_3 are parametrized by some non-abelian group with underlying set $X \times K$, and both the group operation and the commutator relations involve the maps h and π . If L is finite-dimensional over its center, of degree d , and X is finite-dimensional over L , then these Moufang quadrangles arise from algebraic groups. If the involution is of the second kind, they are of absolute type ${}^2\text{A}_\ell$. If the involution is of the first kind, they are of absolute type C_ℓ , ${}^1\text{D}_\ell$ or ${}^2\text{D}_\ell$.

²Again, the situation is slightly more complicated in characteristic two, but we omit the details.

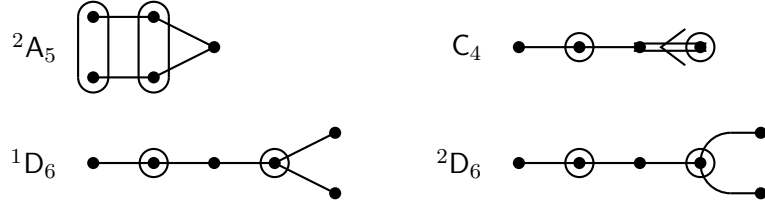


Figure 11: Some Moufang quadrangles of pseudo-quadratic form type

Moufang quadrangles of type E_6 , E_7 and E_8 Moufang quadrangles of type E_6 , E_7 and E_8 are exceptional, and always arise from algebraic groups. The explicit construction of these Moufang quadrangles is very complicated, and one of the goals of our paper is precisely to get a better understanding of these exceptional quadrangles. We refer to [TW, Chapter 12 and 13] or [W1, Chapter 10], or also to [DV, section 4.3.5], for the precise construction; many of its properties will be captured in the definition of a quadrangular algebra that we will recall in section 3.1 below. In section 7, we will give the necessary details to make the connection with structurable algebras.

The corresponding Tits indices are as follows.

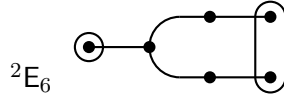


Figure 12: Moufang quadrangles of type E_6

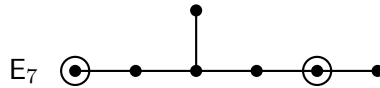


Figure 13: Moufang quadrangles of type E_7

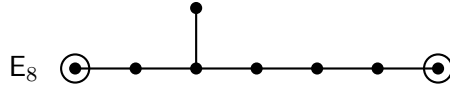


Figure 14: Moufang quadrangles of type E_8

3 Quadrangular algebras, Freudenthal triple systems and structurable algebras

In this section, we assemble some known facts about the different kinds of algebraic structures that we will need, namely quadrangular algebras, Freudenthal triple systems and structurable algebras.

3.1 Quadrangular algebras

In order to try to have a better understanding of the exceptional Moufang quadrangles, i.e. those of type E_6 , E_7 , E_8 and F_4 , Richard Weiss designed an algebraic structure, called a *quadrangular algebra* [W1]. These algebras are a generalization of (some) pseudo-quadratic spaces, where, in some sense, the structure of the underlying skew field is lost, but is replaced by some weaker identities, comparable to the replacement of the associativity by some weaker identities in the definition of an alternative algebra. Unfortunately, the definition involves a large number of maps and identities between these maps, that make it rather difficult to have a deep understanding of the corresponding structures. Moreover, giving an *explicit* construction of the quadrangular algebras related to the exceptional Moufang quadrangles, involves a very delicate way of introducing coordinates and defining these maps in terms of these coordinates. We hope that our understanding in terms of structurable algebras will eventually allow to give an explicit construction avoiding these coordinates, and thereby giving much more insight.

The definition of quadrangular algebras is significantly simpler over fields of characteristic different from 2; since this is the only case we will be dealing with in this paper, we will restrict to this case, and we refer the interested reader to [W1] for the general definition.

Definition 3.1. A *quadrangular algebra* of characteristic different from 2 is an 8-tuple $(K, L, q, 1, X, \cdot, h, \theta)$, where

- (i) K is a commutative field with $\text{char}(K) \neq 2$,
- (ii) L is a K -vector space,
- (iii) q is an anisotropic quadratic form from L to K ,
- (iv) $1 \in L$ is a *base point* for q , i.e. an element such that $q(1) = 1$,
- (v) X is a non-trivial K -vector space,
- (vi) $(a, v) \mapsto a \cdot v$ is a map from $X \times L$ to X (usually denoted simply by juxtaposition),
- (vii) h is a map from $X \times X$ to L , and
- (viii) θ is a map from $X \times L$ to L ,

satisfying the following axioms, where

$$\begin{aligned} f: L \times L &\rightarrow K: (a, b) \mapsto f(a, b) := q(a + b) - q(a) - q(b); \\ \sigma: L &\rightarrow L: v \mapsto f(1, v)1 - v; \\ v^{-1} &:= v^\sigma / q(v). \end{aligned}$$

(A1) The map \cdot is K -bilinear.

(A2) $a \cdot 1 = a$ for all $a \in X$.

(A3) $(av)v^{-1} = a$ for all $a \in X$ and all $v \in L^*$.

(B1) h is K -bilinear.

(B2) $h(a, bv) = h(b, av) + f(h(a, b), 1)v$ for all $a, b \in X$ and all $v \in L$.

(B3) $f(h(av, b), 1) = f(h(a, b), v)$ for all $a, b \in X$ and all $v \in L$.

(C) $\theta(a, v) = \frac{1}{2}h(a, av)$.

(D1) Let $\pi(a) = \theta(a, 1)$ for all $a \in X$. Then $a\theta(a, v) = (a\pi(a))v$ for all $a \in X$ and all $v \in L$.

(D2) $\pi(a) \equiv 0 \pmod{K}$ if and only if $a = 0$ (where K has been identified with its image under the map $t \mapsto t \cdot 1$ from K to L).

Moreover, we define a map $g: X \times X \rightarrow K$ by

$$g(a, b) := \frac{1}{2}f(h(a, b), 1)$$

for all $a, b \in X$.

Quadrangular algebras have been classified by Richard Weiss, and over fields of characteristic different from two, the result can be summarized as follows.

Theorem 3.2 ([W1]). *Let Ω be a quadrangular algebra over a field K with $\text{char}(K) \neq 2$. Then either Ω is an anisotropic pseudo-quadratic space over a quadratic extension E/K or over a quaternion division algebra Q/K equipped with the standard involution, or Ω is of type E_6, E_7 or E_8 .*

Proof. See [W1, Theorem 3.1, Theorem 3.2 and Proposition 3.14]. \square

Remark 3.3. If q is a quadratic form from L to K , with base point $1 \in L$, then the *Clifford algebra of q with basepoint 1* is defined as

$$C(q, 1) := T(L) / \langle u \otimes u^\sigma - q(u) \cdot 1 \rangle,$$

where $T(L)$ is the tensor algebra of L , and where σ is defined as in Definition 3.1. In [TW, (12.51)] it is shown that $C(q, 1) \cong C_0(q)$, the even Clifford algebra of q . The notion of a Clifford algebra with base point was introduced by Jacobson and McCrimmon in [JM]; see also [TW, Chapter 12] for more details. Since q is anisotropic, axioms (A1)–(A3) say precisely that X is a $C(q, 1)$ -module.

Remark 3.4. The definition of g we have used is as in [W1]. This is *not* the same definition as in [TW, Chapter 13], where $g(a, b) = \frac{1}{2}f(h(b, a), 1) = -\frac{1}{2}f(h(a, b), 1)$. See also [W1, Remark (viii), p. 7].

Remark 3.5. The definition of quadrangular algebras over fields K with $\text{char}(K) = 2$ involves four more axioms (C1)–(C4) which replace the axiom (C) in Definition 3.1 above (and define the map g in a different way). One of these axioms, (C4), is considerably more complicated than the other axioms, but it can be shown that the axioms (C1)–(C4) are superfluous over fields K with $\text{char}(K) \neq 2$ in the sense that axiom (C) implies (C1)–(C4). See [W1, Remark 4.8] for more details.

We will use the following formulas in the sequel.

Theorem 3.6. *Let $(K, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra, with $\text{char}(K) \neq 2$. For all $a, b \in X$ and all $u, v \in L$ we have that*

- (i) $h(a, b) = -h(b, a)^\sigma$,
- (ii) $f(h(a, bv), 1) = f(h(a, b), v^\sigma)$,
- (iii) $(au)v = -(av^\sigma)u^\sigma + af(u, v^\sigma)$,
- (iv) $h(a\pi(a), b) + \theta(a, h(a, b)) = 0$,
- (v) $\theta(av, w) = \theta(a, w^\sigma)^\sigma q(v) - f(w, v^\sigma)\theta(a, v)^\sigma + f(\theta(a, v), w^\sigma)v^\sigma$.

Proof. Identities (i)–(iii) are precisely [W1, (3.6), (3.7) and (3.8)]. Identity (iv) is identity (e) in the proof of [TW, (13.67)]; the proof holds without any change in the pseudo-quadratic case as well. Identity (v) is precisely axiom (C4) in [W1, Definition 1.17], taking into account that the map ϕ occurring in this axiom is trivial by [W1, Proposition 4.5]. \square

3.2 Freudenthal triple systems

Definition 3.7. A *Freudenthal triple system* (V, b, t) is a vector space V over a field K of characteristic not 2 or 3, endowed with a trilinear symmetric product

$$t: V \times V \times V \rightarrow V: (x, y, z) \mapsto t(x, y, z) =: xyz$$

and a skew symmetric bilinear form

$$b: V \times V \rightarrow K: (x, y) \mapsto b(x, y) =: \langle x, y \rangle$$

such that

- (i) the map $(x, y, z, w) \mapsto \langle x, yzw \rangle$ is a nonzero symmetric 4-linear form;
- (ii) $(xxx)xy = \langle y, x \rangle xxx + \langle y, xxx \rangle x \quad \forall x, y \in V$.

When it is clear which triple product and skew symmetric form are considered, we do not explicitly mention b and t , but we use juxtaposition and $\langle \cdot, \cdot \rangle$ instead.

Definition 3.8. Two Freudenthal triple systems (V, b, t) , (V', b', t') over a field K are *similar* if there exists a K -vector space isomorphism $\psi : V \rightarrow V'$ and $\lambda \in K^*$ such that

$$t'(\psi(x), \psi(y), \psi(z)) = \lambda \psi(t(x, y, z)).$$

In [F, Lemma 6.6] it is proven that this condition is equivalent with

$$\begin{cases} b'(\psi(x), \psi(y)) = \lambda b(x, y) & \text{and} \\ b'(\psi(x), t'(\psi(x), \psi(x), \psi(x))) = \lambda^2 b(x, t(x, x, x)). \end{cases}$$

The map ψ is then called a *similarity* with *multiplier* λ . We say that two Freudenthal triple systems are *isometric* if they are similar with $\lambda = 1$; in this case ψ is called an *isometry*.

Definition 3.9. Let V be a Freudenthal triple system.

- (i) An element $u \in V \setminus \{0\}$ is called *strictly regular* if $uVV \subseteq Ku$.
- (ii) A pair of strictly regular elements u_1, u_2 is called *supplementary* if $\langle u_1, u_2 \rangle = 1$.
- (iii) V is called *reduced* if it contains a strictly regular element.
- (iv) V is called *simple* if it does not contain a proper ideal, i.e. a subspace $I \neq 0, V$ such that $IVV \subseteq I$.

More details can be found in [F]. We mention a few results that we will use later.

Lemma 3.10. *If the map $x \mapsto \langle x, xxx \rangle$ is anisotropic, the Freudenthal triple system is not reduced and simple.*

Proof. Suppose $u \in V$ is strictly regular, then $uuu = ku$ for some $k \in K$, so

$$\langle u, uuu \rangle = k \langle u, u \rangle = 0.$$

This implies that $u = 0$, so the Freudenthal triple system is not reduced.

In [F] it is shown that a Freudenthal triple system is simple if and only if its bilinear form is nondegenerate. This is clearly the case, since for every $x \in V \setminus \{0\}$, we have $\langle x, y \rangle \neq 0$ for $y = xxx$. \square

Lemma 3.11 ([F, Corollary 3.4]). *A simple Freudenthal triple system V is reduced if and only there exists $x \in V$ such that $\langle x, xxx \rangle = 12k^2$ for $k \in K^*$.*

If this is the case, then

$$u_1 = \frac{1}{2}x + \frac{1}{12k}xxx, \quad u_2 = -\frac{1}{2k}x + \frac{1}{12k^2}xxx$$

is a pair of supplementary strictly regular elements.

3.3 Structurable algebras

Structurable algebras have been introduced by B. Allison [A1] and have been used in the construction of non-split exceptional simple Lie algebras, see for example [A2].

Definition 3.12. A *structurable algebra* over a field K of characteristic not 2 or 3 is a unital, not necessarily associative K -algebra with involution³ $(\mathcal{A}, \bar{})$ such that

$$[V_{x,y}, V_{z,w}] = V_{\{x,y,z\},w} - V_{z,\{y,x,w\}} \quad (3.1)$$

for $x, y, z, w \in \mathcal{A}$ where $V_{x,y}z := \{x, y, z\} := (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$.

For all $x, y, z \in \mathcal{A}$, we write $U_{x,y}z := V_{x,z}y$ and $U_xy := U_{x,x}y$. We will refer to the maps $V_{x,y} \in \text{End}_K(\mathcal{A})$ as *V-operators*, and to the maps $U_{x,y} \in \text{End}_K(\mathcal{A})$ and $U_x \in \text{End}_K(\mathcal{A})$ as *U-operators*.

Structurable algebras are generalizations of both associative algebras with involution and Jordan algebras. Indeed, the class of structurable algebras with trivial involution is exactly equal to the class of Jordan algebras (in characteristic different from 2).

If $(\mathcal{A}, \bar{})$ is a structurable algebra, then $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$ for

$$\mathcal{H} = \{h \in \mathcal{A} \mid \bar{h} = h\} \quad \text{and} \quad \mathcal{S} = \{s \in \mathcal{A} \mid \bar{s} = -s\}.$$

The dimension of \mathcal{S} is called the *skew-dimension* of \mathcal{A} . Structurable algebras of skew-dimension 0 are exactly the Jordan algebras.

Definition 3.13. An element $x \in \mathcal{A}$ is called *conjugate invertible* if it has a *conjugate inverse*, i.e. an element \hat{x} such that $V_{\hat{x},x} = V_{x,\hat{x}} = \text{id}$. If an element is conjugate invertible, then it has a unique conjugate inverse. A structurable algebra is a (*conjugate*) *division algebra* if every non-zero element is conjugate invertible.

Although it is possible to define and study isomorphisms of structurable algebras, it turns out that it is better to allow the unit element 1 to be mapped to a different element. This idea is encapsulated in the notion of an isotopy.

Definition 3.14. Two structurable algebras $(\mathcal{A}, \bar{})$ and $(\mathcal{A}', \bar{})$ over a field K are *isotopic* if there exists two K -vector space isomorphisms $\psi, \chi : \mathcal{A} \rightarrow \mathcal{A}'$ such that

$$\psi(V_{x,y}z) = V_{\psi(x),\chi(y)}\psi(z) \quad \forall x, y, z \in \mathcal{A}.$$

The map ψ is then called an *isotopy* between $(\mathcal{A}, \bar{})$ and $(\mathcal{A}', \bar{})$.

³An involution is a K -linear map of order 2 such that $\overline{\overline{xy}} = \overline{y}\overline{x}$.

- Remark 3.15.** (i) If ψ is an isotopy, the map χ is entirely determined by the map ψ .
- (ii) If ψ maps the identity of \mathcal{A} to the identity of \mathcal{A}' , ψ is an isomorphism of structurable algebras.
- (iii) If $(\mathcal{A}', -)$ is isotopic to $(\mathcal{A}, -)$, then there exists a conjugate invertible $u \in \mathcal{A}'$ such that $(\mathcal{A}', -)^{\langle u \rangle}$ is isomorphic to $(\mathcal{A}, -)$. For a description of $(\mathcal{A}', -)^{\langle u \rangle}$ we refer to [AF1, p. 188] or [A3, section 2].

Remark 3.16. Central simple structurable algebras over fields of characteristic different from 2, 3 and 5, have been classified. They consist of six classes:

- (1) associative algebras with involution,
- (2) Jordan algebras,
- (3) structurable algebras constructed from a hermitian form over an associative algebra with involution (see section 6),
- (4) forms of structurable matrix algebras (see Example 3.21),
- (5) forms of tensor products of composition algebras,
- (6) an exceptional 35-dimensional case.

For a proof we refer to [S, Theorem 3.8].

In an associative algebra with involution, an element is conjugate invertible if and only if it is invertible in the usual associative sense. If x is invertible, its conjugate inverse is equal to \bar{x}^{-1} .

In a Jordan algebra, an element is conjugate invertible if and only if it is invertible in the usual Jordan sense. If x is invertible, its conjugate inverse is equal to its Jordan inverse x^{-1} .

3.3.1 Structurable algebras of skew-dimension one

Structurable algebras of skew-dimension one are close to Jordan algebras. Although they are in general not power-associative, concepts from Jordan theory can be adapted to this class of structurable algebras; see [A3]. It is of particular interest to us that one can give each structurable algebra of skew-dimension one the structure of a simple Freudenthal triple system.

We give some examples of structurable algebras of skew-dimension one by considering the classification in Remark 3.16: the algebras in case (4) always have skew-dimension one; those in cases (1) and (3) have skew-dimension one if and only if the associative algebra with involution has skew-dimension one. The structurable algebras in the remaining cases never have skew-dimension one, except for forms of $E \otimes_K K$ with E a quadratic field extension of the field K .

In this section $(\mathcal{A}, -)$ is always a structurable algebra of skew-dimension one. Such an algebra is always central simple.

We fix a non-zero element $s_0 \in \mathcal{S}$, so $\mathcal{S} = Ks_0$. One can show that $s_0^2 = \mu 1$ for some $\mu \in K^*$ and that $s_0(s_0x) = (xs_0)s_0 = \mu x$, for all $x \in \mathcal{A}$.

Theorem 3.17 ([AF1, Proposition 2.8]). *Let $(\mathcal{A}, \bar{})$ be a structurable algebra of skew-dimension one and let $s_0 \in \mathcal{S}$. The following triple product and bilinear form give \mathcal{A} the structure of a simple Freudenthal triple system:*

$$\begin{aligned}\langle x, y \rangle 1 &= (x\bar{y} - y\bar{x})s_0, \\ yzw &= 2\{y, s_0z, w\} - \langle z, w \rangle y - \langle z, y \rangle w - \langle y, w \rangle z.\end{aligned}$$

Remark 3.18. (i) If we would choose another generator for \mathcal{S} , then we would get a Freudenthal triple system that is a scalar multiple of the one we started with.

(ii) In structurable algebras there exists a generalization of the generic norm in a Jordan algebra, called the *conjugate norm*. For an exact definition see [AF2].

For algebras of skew-dimension one, the conjugate norm, denoted by ν , is exactly $\frac{1}{12\mu}\langle x, xxx \rangle$. Indeed, one can easily verify that the norm in [AF2, Prop. 5.4] and the definition of ν in [A3, Par. 1] are given by the same formulas. The quartic map ν is independent of the choice of s_0 .

In structurable algebras of skew-dimension one there is an easy way to write down the conjugate inverse of an element.

Theorem 3.19 ([AF1, Prop. 2.11]). *Let $(\mathcal{A}, \bar{})$ be a structurable algebra of skew-dimension one and let $s_0 \in \mathcal{S}$. Then $x \in \mathcal{A}$ is conjugate invertible (see Definition 3.13) if and only if $\nu(x) \neq 0$. When this is the case, we have*

$$\hat{x} = -\frac{1}{3\mu\nu(x)}s_0\{x, s_0x, x\}.$$

On a structurable algebra, we have the notion of similarity; on a Freudenthal triple system, we have the notion of isotopy. Theorem 3.17 tells us that a structurable algebra of skew-dimension one is also a Freudenthal triple system. The following lemma states that in this case the notions of isotopy and similarity coincide.

Lemma 3.20 ([G, Proposition 4.11]). *Let $(\mathcal{A}, \bar{})$ and $(\mathcal{A}', \bar{})$ be structurable algebras of skew-dimension one. Consider the corresponding Freudenthal triple systems as in Theorem 3.17. Then \mathcal{A} and \mathcal{A}' are similar as Freudenthal triple systems if and only if they are isotopic as structurable algebras.*

Proof. In [G], all Freudenthal triple systems that are considered are 56-dimensional, but the proof remains valid in arbitrary dimension. \square

We will now discuss an important class of structurable algebras of skew-dimension one, which we will call structurable matrix algebras.

Example 3.21 ([A3, Example 1.9]). We give a short overview of how this class of algebras and the corresponding Freudenthal triple systems are constructed. For more details, see [A3, Example 1.9].

Let J be a Jordan algebra over a field K constructed from an admissible cubic form N with base point, or a Jordan algebra constructed from a non-degenerate quadratic form. (For definitions of these Jordan algebras we refer to [M, Chapter II.3.3 and II.4]; an admissible cubic form is the same as a Jordan cubic form in [M, II.4.3].)

The Jordan algebra J is equipped with a non-degenerate trace form T and a symmetric sharp product \times (sometimes denoted by \sharp).

We define the *structurable matrix algebras* as follows. Fix a constant $\eta \in K$, and define

$$\mathcal{A} = \left\{ \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \mid k_1, k_2 \in K, j_1, j_2 \in J \right\}.$$

For $k_1, k_2, k'_1, k'_2 \in K$, $j_1, j_2, j'_1, j'_2 \in J$, define the involution and multiplication as follows:

$$\overline{\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}} = \begin{pmatrix} k_2 & j_1 \\ j_2 & k_1 \end{pmatrix},$$

$$\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \begin{pmatrix} k'_1 & j'_1 \\ j'_2 & k'_2 \end{pmatrix} = \begin{pmatrix} k_1 k'_1 + \eta T(j_1, j'_2) & k_1 j'_1 + k'_2 j_1 + \eta(j_2 \times j'_2) \\ k'_1 j_2 + k_2 j'_2 + j_1 \times j'_1 & k_2 k'_2 + \eta T(j_2, j'_1) \end{pmatrix}.$$

We denote this structurable matrix algebra by $M(J, \eta)$.

Now let $s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; then the Freudenthal triple system defined in Theorem 3.17 has bilinear product

$$\left\langle \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}, \begin{pmatrix} k'_1 & j'_1 \\ j'_2 & k'_2 \end{pmatrix} \right\rangle = k_1 k'_2 - k_2 k'_1 + \eta T(j_1, j'_2) - \eta T(j_2, j'_1),$$

and the conjugate norm is given by

$$\nu \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} = 4k_1 \eta N(j_1) + 4k_2 \eta^2 N(j_2) - 4\eta^2 T(j_1^\sharp, j_2^\sharp) + (\eta T(j_1, j_2) - k_1 k_2)^2.$$

The following theorem shows that each structurable algebra of skew-dimension one is isomorphic to a matrix algebra or becomes isomorphic to a matrix algebra after adjoining $\sqrt{\mu}$ to the base field.

Theorem 3.22 ([AF1, Prop. 4.5]). *Let $(\mathcal{A}, ^-)$ be a structurable algebra of skew-dimension one, and let $s_0 \in \mathcal{S}$ with $s_0^2 = \mu 1$. Then $(\mathcal{A}, ^-)$ is isomorphic to a matrix algebra $M(J, \eta)$ if and only if μ is a square in K .*

4 Quadrangular algebras and Freudenthal triple systems

We show that each quadrangular algebra defined over a field of characteristic not 2 or 3 can be given the structure of a Freudenthal triple system. It turns out that the maps $x \mapsto x\pi(x)$ and $x \mapsto q(\pi(x))$, which play an important role in the structure of (the non-abelian rank one residue of) the quadrangles of type E_6, E_7 and E_8 , also play an important role in the Freudenthal triple system.

Theorem 4.1. *Let $(K, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra over a field K with $\text{char}(K) \neq 2, 3$, and let π be as in Definition 3.1. Then X is a Freudenthal triple system with triple product*

$$xyz := \frac{1}{2}(x(h(y, z) + h(z, y)) + y(h(x, z) + h(z, x)) + z(h(x, y) + h(y, x)))$$

and skew symmetric bilinear form $\langle x, y \rangle := g(x, y)$, for all $x, y, z \in X$. This Freudenthal triple system is simple and not reduced.

Furthermore we have that xyz is the linearization⁴ of $x\pi(x)$ and $\langle x, yzw \rangle$ is the linearization of $-\frac{1}{2}q(\pi(x))$.

In particular, $xxx = 6x\pi(x)$ and $\langle x, xxx \rangle = -12q(\pi(x))$.

Proof. It is clear that the triple product is symmetric and trilinear. It follows from (3.6) that g is skew symmetric and bilinear, and that $\langle x, yzw \rangle$ is linear in its four variables. Since $\pi(x) = \frac{1}{2}h(x, x)$ we have that $xxx = 6x\pi(x)$, so xyz is the linearization of $x\pi(x)$. To prove the first axiom of Definition 3.7 we expand $\langle x, yzw \rangle$, and we find

$$\begin{aligned} g(x, yzw) &= \frac{1}{2}f(h(x, yzw), 1) \\ &= \frac{1}{4}(f(h(x, w(h(y, z) + h(z, y))), 1) + f(h(x, y(h(w, z) + h(z, w))), 1) \\ &\quad + f(h(x, z(h(w, y) + h(y, w))), 1)) \\ &= \frac{1}{4}(f(h(x, w), \overline{h(y, z) + h(z, y)}) + f(h(x, y), \overline{h(w, z) + h(z, w)}) \\ &\quad + f(h(x, z), \overline{h(w, y) + h(y, w)})) \\ &= -\frac{1}{4}(f(h(x, w), h(y, z) + h(z, y)) + f(h(x, y), h(w, z) + h(z, w)) \\ &\quad + f(h(x, z), h(w, y) + h(y, w))). \end{aligned}$$

Therefore $\langle x, yzw \rangle$ is indeed symmetric and linear in its four variables. When we put $x = y = z = w$, this expression equals $-12q(\pi(x))$. Thus it is the

⁴Of course, we mean that the map from $X \times X \times X$ to X mapping (x, y, z) to xyz is the linearization of the cubic map from X to X mapping x to $x\pi(x)$, but our slight abuse of language should not cause any confusion. Note that we use the convention that the linearization of a homogeneous map A of degree n is the symmetric n -linear map $B(x_1, \dots, x_n)$ such that $B(x, \dots, x) = n! \cdot A(x)$.

linearization of $-\frac{1}{2}q(\pi(x))$. This map is non-zero since both q and π are anisotropic.

In order to establish the second axiom, we show that

$$(x\pi(x))xy = \frac{1}{2}(f(h(y, x), 1)x\pi(x) + f(h(y, x\pi(x)), 1)x). \quad (4.1)$$

We expand the left side of this identity, and we get

$$\begin{aligned} (x\pi(x))xy &= \frac{1}{2}\left(x\pi(x)(h(x, y) + h(y, x)) + x(h(x\pi(x), y) + h(y, x\pi(x)))\right) \\ &\quad + y(h(x, x\pi(x)) + h(x\pi(x), x)). \end{aligned}$$

It follows from Theorem 3.6(iv) that the third term is zero. To reduce the two other terms we use

$$h(y, z) + h(z, y) = h(y, z) - \overline{h(y, z)} = 2h(y, z) - f(h(y, z), 1)1,$$

and we get

$$\begin{aligned} (x\pi(x))xy &= \frac{1}{2}\left(2x\pi(x)h(x, y) - x\pi(x)f(h(x, y), 1)\right. \\ &\quad \left.+ 2xh(x\pi(x), y) - xf(h(x\pi(x), y), 1)\right) \\ &= x(\theta(x, h(x, y)) + h(x\pi(x), y)) \\ &\quad + \frac{1}{2}(x\pi(x)f(h(y, x), 1) + xf(h(y, x\pi(x)), 1)), \end{aligned}$$

where we have used (D1). It follows from Theorem 3.6(iv) that the first term is zero, establishing (4.1).

As $\langle x, xxx \rangle = -12q(\pi(x))$ is anisotropic it follows from Lemma 3.10 that the Freudenthal triple system we obtained is simple and not reduced. \square

Remark 4.2. One should be careful not to confuse between the notation for the triple product xyz for $x, y, z \in X$ and the map $X \times L \rightarrow X$, defined in Definition 3.1, which have also denoted by juxtaposition. However, as there is no multiplication defined on X , we will never write xy with $x, y \in X$, so our notation is always unambiguous.

On the other hand, for $x \in X$ and $v, w \in L$ the term xvw could be interpreted in two ways. However without brackets this will always denote the triple product, whereas with brackets $(xv)w$ this denotes applying the $X \times L \rightarrow X$ map two successive times.

In the next theorem we show that the triple product behaves well with respect to the $C(q, 1)$ -module structure on X (see Remark 3.3).

Theorem 4.3. *For $x, y, z \in X$ and $v \in L \setminus \{0\}$ we have that*

$$(xyz)v = \frac{(xv)(yv)(zv)}{q(v)}.$$

Proof. It is enough to show that this identity holds for $x = y = z$, since the general result then follows by linearizing. Thus we have to show that

$$(x\pi(x))v = \frac{(xv)\pi(xv)}{q(v)}.$$

This follows from [W2, Theorem 3.18], since $(x\pi(x))v = x\theta(x, v)$; the map ϕ occurring in that formula is identically zero for fields of characteristic not 2. (In *loc. cit.*, only quadrangular algebras of type E_6, E_7 and E_8 are considered, but this proof is also valid for pseudo-quadratic spaces.) \square

5 Structurable algebras on arbitrary quadrangular algebras

In this main section of the paper, we will show that every quadrangular algebra in characteristic not 2 or 3 gives rise to a family of isotopic structurable algebras, in such a way that certain concepts from the theory of the quadrangular algebras and from the theory of structurable algebras coincide.

Theorem 5.1. *Let $(K, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra with $\text{char}(K) \neq 2, 3$. Then there exists a family of pairwise isotopic structurable algebras on X , such that each algebra \mathcal{A} in this family satisfies the following properties:*

- (i) \mathcal{A} has skew-dimension one,
- (ii) \mathcal{A} is a division algebra,
- (iii) there is a skew-symmetric element $s_0 \in \mathcal{S}$ such that $U_x(s_0x) = 3x\pi(x)$ for all $x \in \mathcal{A}$,
- (iv) the conjugate norm of \mathcal{A} is a scalar multiple of $q \circ \pi$,
- (v) the conjugate inverse in \mathcal{A} behaves as the inverse in the Moufang quadrangle; see section 5.2 below.

Theorem 5.1 is a consequence of Theorems 4.1, 5.2 and 5.4 below.

5.1 The main construction

In order to define a structurable algebra on X , we make use of the Freudenthal triple system that we have described in the previous section.

Theorem 5.2. *Let (V, t, b) be a simple Freudenthal triple system. There exists a structurable algebra $(\mathcal{A}, \bar{})$ of skew-dimension one, such that (V, t, b) is isometric to $(\mathcal{A}, \bar{})$, considered as a Freudenthal triple system as in Theorem 3.17.*

Proof. We can apply the construction in [G, Lemma 4.15]. In *loc. cit.*, a Freudenthal triple system is defined to be of dimension 56; it is however easily verified that this construction can be carried out for simple Freudenthal triple systems of arbitrary dimension. \square

We want to describe the structurable algebra, obtained by combining Theorems 4.1 and 5.2, in a more detailed way than in [G, Lemma 4.15]. In order to do this we have to make the construction much more explicit. The structurable algebra constructed in Theorem 5.2 is obtained in three steps:

- Step 1.* We tensor the simple Freudenthal triple system X with a quadratic field extension Δ such that it becomes reduced.
- Step 2.* We apply the proof of [F, Theorem 5.1] to construct a structurable matrix algebra that is isometric to $X \otimes_K \Delta$.
- Step 3.* We use the methods from [G, Lemma 4.15] to apply Galois descent and find a structurable algebra that is isometric to X .

Construction 5.3. Let $\Omega = (K, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra with $\text{char}(K) \neq 2, 3$, and consider X as a simple non-reduced Freudenthal triple system as in Theorem 4.1.

Step 1: Extending scalars to make X reduced.

To reduce X we use Lemma 3.11. For all $x \in X$, we have $\langle x, xxx \rangle = 12(-q(\pi(x)))$. Since X is not reduced, $-q(\pi(x))$ is never a square in K .

We fix an arbitrary $a \in X^*$ and define $\delta := \sqrt{-q(\pi(a))}$ in the algebraic closure of K , so that $\Delta = K(\delta)$ is a quadratic field extension of K ; let ι be the non-trivial element of $\text{Gal}(\Delta/K)$.

We now linearly extend the trilinear product and the bilinear form on X to $X \otimes_K \Delta$. This makes $X \otimes_K \Delta$ into a Freudenthal triple system. By our choice of Δ , the Freudenthal triple system $X \otimes_K \Delta$ is reduced. By Lemma 3.11 and Theorem 4.1,

$$u'_1 = \frac{1}{2} \left(a + \frac{a\pi(a)}{\delta} \right), \quad u'_2 = \frac{1}{2\delta} \left(-a + \frac{a\pi(a)}{\delta} \right)$$

form a supplementary pair of strictly regular elements.

Step 2: Construction of a structurable matrix algebra isometric to $X \otimes_K \Delta$.

We point out that if we say that a structurable matrix algebra $M(J, \eta)$ is isometric to $X \otimes_K \Delta$, we mean that the Freudenthal triple system $M(J, \eta)$, defined by the formulas for $\langle \cdot, \cdot \rangle$ and ν in Example 3.21, is isometric to the Freudenthal triple system $X \otimes_K \Delta$.

In order to construct a structurable matrix algebra that is isometric to $X \otimes_K \Delta$, we have to construct a Jordan algebra over Δ . We proceed as

in [F], but we slightly modify the construction which is presented there. We only give the necessary ingredients, referring the reader to *loc. cit.* for more details.

For $\epsilon \in \{1, -1\}$, we let

$$M_\epsilon := \{x \in X \otimes_K \Delta \mid u'_1 u'_2 x = \epsilon x\}.$$

As in *loc. cit.*, we will define a Jordan algebra on the vector space M_1 . This Jordan algebra will be constructed either from a quadratic form or from an admissible cubic form.

If the expression $g(u'_1, y\pi(y))$ is identically zero for $y \in M_1$, then there is a quadratic form Q on M_1 making M_1 into a Jordan algebra; in this case we define $N = 0$ and $\lambda = 1$.

On the other hand, if there exists an $e \in M_1$ such that $g(u'_1, e\pi(e)) \neq 0$, then

$$N(x) := \frac{g(u'_1, x\pi(x))}{g(u'_1, e\pi(e))}$$

is an admissible cubic form on M_1 with base point e , making M_1 into a Jordan algebra, and we let

$$\lambda := \frac{1}{2}g(u'_1, e\pi(e)) \in \Delta.$$

It is shown in *loc. cit.* that in both cases, $X \otimes_K \Delta$ is isometric to the structurable matrix algebra $M(M_1, \lambda)$ (as defined in Example 3.21). However, we prefer to slightly modify the construction so that $X \otimes_K \Delta$ becomes isometric to $M(M_1, 1)$. One obvious way to do this is to redefine the pair of strictly regular elements as $u_1 = \lambda^{-1}u'_1$ and $u_2 = \lambda u'_2$, so that

$$u_1 = \frac{1}{2\lambda} \left(a + \frac{a\pi(a)}{\delta} \right), \quad u_2 = \frac{\lambda}{2\delta} \left(-a + \frac{a\pi(a)}{\delta} \right);$$

then $X \otimes_K \Delta$ will indeed be isometric to $M(M_1, 1)$. Note that the spaces M_ϵ are unchanged by replacing u'_1 and u'_2 by u_1 and u_2 , respectively.

In *loc. cit.* it is shown that $X \otimes_K \Delta = \Delta u_1 \oplus \Delta u_2 \oplus M_1 \oplus M_{-1}$, and that there exists an isomorphism $t: M_1 \rightarrow M_{-1}$. This allows us to explicitly write down the isometry ψ between $X \otimes_K \Delta$ and $M(M_1, 1)$:

$$\begin{aligned} \psi : \Delta u_1 \oplus \Delta u_2 \oplus M_1 \oplus M_{-1} &\rightarrow \begin{pmatrix} \Delta & M_1 \\ M_1 & \Delta \end{pmatrix} : \\ d_1 u_1 + d_2 u_2 + j_1 + t(j_2) &\mapsto \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix}, \end{aligned}$$

for all $d_1, d_2 \in \Delta$ and all $j_1, j_2 \in M_1$. So we obtain a structurable algebra $M(M_1, 1)$ that is defined over Δ and isometric to $X \otimes_K \Delta$.

Step 3: Galois descent.

Our next step is to apply Galois descent to obtain a structurable algebra over K isometric to X . We follow the ideas of [G, Lemma 4.15], but we use a more explicit approach in order to obtain exact formulas.

Let $\tilde{\eta}$ be the extension of ι to $X \otimes_K \Delta$ given by

$$\tilde{\eta}(x \otimes d) := x \otimes \iota(d).$$

Since the fixed point set of $\tilde{\eta}$ in $X \otimes_K \Delta$ is X , we determine how this map acts on $M(M_1, 1)$. As $\tilde{\eta}(xyz) = \tilde{\eta}(x)\tilde{\eta}(y)\tilde{\eta}(z)$, the map $\tilde{\eta}$ is an isometry of the Freudenthal triple system. We have $\tilde{\eta}(u_1) = \frac{-\delta}{N(\lambda)}u_2$, and it follows from $\tilde{\eta}(u_1u_2x) = -u_1u_2\tilde{\eta}(x)$ that $x \in M_{\pm 1}$ if and only if $\tilde{\eta}(x) \in M_{\mp 1}$.

The explicit formula for $\tilde{\eta}$ is given by

$$\begin{aligned} \tilde{\eta}(d_1u_1 + d_2u_2 + j_1 + t(j_2)) \\ &= \iota(d_1)\frac{-\delta}{N(\lambda)}u_2 + \iota(d_2)\frac{N(\lambda)}{\delta}u_1 + \tilde{\eta}(j_1) + \tilde{\eta}(t(j_2)) \\ &= \iota(d_2)\frac{N(\lambda)}{\delta}u_1 + \iota(d_1)\frac{-\delta}{N(\lambda)}u_2 + \tilde{\eta}(t(j_2)) + t(t^{-1}(\tilde{\eta}(j_1))), \end{aligned}$$

for all $d_1, d_2 \in \Delta$ and all $j_1, j_2 \in M_1$. Since $\tilde{\eta}(t(j_2)), t^{-1}(\tilde{\eta}(j_1)) \in M_1$, we can translate this into matrix notation using ψ , and we get

$$\tilde{\eta} : \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{N(\lambda)}{\delta}\iota(d_2) & \tilde{\eta}(t(j_2)) \\ t^{-1}(\tilde{\eta}(j_1)) & \frac{-\delta}{N(\lambda)}\iota(d_1) \end{pmatrix}.$$

We denote the Freudenthal triple system on $\mathcal{A} := M(M_1, 1)$ from Example 3.21 by (\mathcal{A}, b, t) ; it follows that $\tilde{\eta}$ is an isometry of (\mathcal{A}, b, t) . It is important to note, however, that $\tilde{\eta}$ is in general *not* an algebra automorphism of \mathcal{A} , and the fixed points of $\tilde{\eta}$ in \mathcal{A} do *not* form a structurable algebra.

Following [G], we consider the structurable algebra $\mathcal{A}' := M(M_1, \frac{\delta}{N(\lambda)})$; denote the corresponding Freudenthal triple system by (\mathcal{A}', b', t') . We now modify this Freudenthal triple system once more. Let

$$s'_0 = \frac{N(\lambda)}{\delta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and consider the Freudenthal triple system associated to \mathcal{A}' with respect to s'_0 as in Theorem 3.17. Then we obtain the Freudenthal triple system (\mathcal{A}', b'', t'') , where

$$b'' = \frac{N(\lambda)}{\delta}b' \quad \text{and} \quad t'' = \frac{N(\lambda)}{\delta}t'.$$

The map

$$\tilde{f} : \mathcal{A} \rightarrow \mathcal{A}' : \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{\delta}{N(\lambda)} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix}$$

is an isometry from (\mathcal{A}, b, t) to (\mathcal{A}', b'', t'') . Now consider the map $\tilde{\pi} := \tilde{f} \tilde{\eta} \tilde{f}^{-1} : \mathcal{A}' \rightarrow \mathcal{A}'$, which is explicitly given by

$$\tilde{\pi} : \begin{pmatrix} d_1 & j_1 \\ j_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} \iota(d_2) & \tilde{\eta}(t(j_2)) \\ t^{-1}(\tilde{\eta}(j_1)) & \iota(d_1) \end{pmatrix}.$$

It is now obvious that $\tilde{\pi}$ is an isometry of (\mathcal{A}', b'', t'') . Using some properties of norm similarities of Jordan algebras, one can show that $\tilde{\pi}$ is, in fact, an algebra automorphism of \mathcal{A}' .

It follows that $\mathcal{A}'^{\tilde{\pi}}$, the fixed points of \mathcal{A}' under $\tilde{\pi}$, is a structurable algebra. Considered as Freudenthal triple systems, $\mathcal{A}'^{\tilde{\pi}}$ and $\mathcal{A}'^{\tilde{\eta}}$ are isometric. Since $\mathcal{A}'^{\tilde{\eta}}$ is in turn isometric to X , the map

$$\tau := \tilde{f} \circ \psi : X \rightarrow \mathcal{A}'^{\tilde{\pi}}$$

is an isometry.

We now use this isometry to make X into a structurable algebra isomorphic to $\mathcal{A}'^{\tilde{\pi}}$, by defining the following multiplication and involution:

$$x \star y := \tau^{-1}(\tau(x)\tau(y)) \quad \text{and} \quad \bar{x} := \tau^{-1}(\overline{\tau(x)})$$

for all $x, y \in X$, where the multiplication and involution in the right hand sides are as in Example 3.21 applied on \mathcal{A}' .

We will denote this structurable algebra by

$$X = X(\Omega, a, \lambda),$$

where Ω is the quadrangular algebra we started from, and where $a \in X^*$ and $\lambda \in \Delta$ are as in Step 1 and Step 2, respectively.

[End of Construction 5.3]

We can now explicitly write down the structurable algebra X in terms of the original quadrangular algebra.

Theorem 5.4. *Let $X = X(\Omega, a, \lambda)$ be as above. Let*

$$\begin{aligned} \mathbf{1} &:= \frac{1}{2\delta} \left(\lambda^\sigma \left(a + \frac{a\pi(a)}{\delta} \right) + \lambda \left(-a + \frac{a\pi(a)}{\delta} \right) \right), \\ s_0 &:= \frac{N(\lambda)}{2\delta^2} \left(\lambda^\sigma \left(a + \frac{a\pi(a)}{\delta} \right) - \lambda \left(-a + \frac{a\pi(a)}{\delta} \right) \right), \\ \mu &:= \frac{N(\lambda)^2}{\delta^2}. \end{aligned}$$

Then X is a structurable algebra with zero element $0 \in X$ and identity element $\mathbf{1} \in X$; X has skew-dimension one, and the subspace \mathcal{S} of skew-symmetric elements is generated by s_0 , with $s_0^2 = \mu$. The subspace \mathcal{H} of symmetric elements is

$$\mathcal{H} = \{x \in X \mid \bar{x} = x\} = \{k\mathbf{1} + j + \tilde{\eta}(j) \mid k \in K, j \in M_1\}.$$

Moreover, for all $x, y, z \in X$, we have

$$\begin{aligned} V_{x, s_0 \star y} z &= \frac{1}{2} \left(xh(y, z) + yh(x, z) + zh(y, x) \right), \\ (x \star \bar{y} - y \star \bar{x}) \star s_0 &= g(x, y)\mathbf{1}, \\ \nu(x) &= -\frac{\delta^2}{N(\lambda)^2} q(\pi(x)), \end{aligned}$$

where ν is the conjugate norm of X (see Remark 3.18(ii)). If we make other choices for $a' \in X^*$ and $\lambda' \in \Delta$, then the structurable algebras $X(\Omega, a, \lambda)$ and $X(\Omega, a', \lambda')$ are isotopic.

Proof. By definition, the isometry τ is an isomorphism from the structurable algebra X to the structurable algebra $\mathcal{A}'^{\tilde{\pi}}$, which is known to be of skew-dimension one. In particular, the zero element and the identity element of X are equal to $0 = \tau^{-1}(0)$ and $\mathbf{1} := \tau^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, respectively. Moreover, X has skew-dimension one, and the set \mathcal{S} of skew-symmetric elements is generated by $s_0 := \tau^{-1}(s'_0)$. We now perform some explicit calculations.

Notice that all elements in $\mathcal{A}'^{\tilde{\pi}}$ are of the form

$$\begin{pmatrix} d & j \\ t^{-1}(\tilde{\eta}(j)) & \iota(d) \end{pmatrix}$$

for some $d \in \Delta$ and $j \in M_1$. We compute $\tau^{-1} := \psi^{-1} \circ \tilde{f}^{-1} : \mathcal{A}'^{\tilde{\pi}} \rightarrow X$:

$$\begin{aligned} \begin{pmatrix} d & j \\ t^{-1}(\tilde{\eta}(j)) & \iota(d) \end{pmatrix} &\xrightarrow{\tilde{f}^{-1}} \begin{pmatrix} \frac{N(\lambda)}{\delta} d & j \\ t^{-1}(\tilde{\eta}(j)) & \iota(d) \end{pmatrix} \in \mathcal{A}^{\tilde{\eta}} \\ &\xrightarrow{\psi^{-1}} \frac{N(\lambda)}{\delta} du_1 + \iota(d)u_2 + j + \tilde{\eta}(j) \in X. \end{aligned}$$

Now we can determine

$$\begin{aligned}
\mathbf{1} &= \tau^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{N(\lambda)}{\delta} u_1 + u_2 \\
&= \frac{1}{2\delta} \left(\lambda^\sigma \left(x + \frac{x\pi(x)}{\delta} \right) + \lambda \left(-x + \frac{x\pi(x)}{\delta} \right) \right), \\
s_0 &= \tau^{-1} \begin{pmatrix} \frac{N(\lambda)}{\delta} & 0 \\ 0 & -\frac{N(\lambda)}{\delta} \end{pmatrix} = \frac{N(\lambda)^2}{\delta^2} u_1 - \frac{N(\lambda)}{\delta} u_2 \\
&= \frac{N(\lambda)}{2\delta^2} \left(\lambda^\sigma \left(x + \frac{x\pi(x)}{\delta} \right) - \lambda \left(-x + \frac{x\pi(x)}{\delta} \right) \right), \\
s_0 \star s_0 &= \tau^{-1}(s'_0 s'_0) = \tau^{-1} \left(\frac{N(\lambda)^2}{\delta^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \frac{N(\lambda)^2}{\delta^2} \mathbf{1}.
\end{aligned}$$

We determine how the involution of the structurable algebra \mathcal{A}' acts on X . We have $\overline{x} = \tau^{-1}(\overline{\tau(x)})$, therefore

$$\begin{aligned}
\overline{\frac{N(\lambda)}{\delta} du_1 + \iota(d)u_2 + j + \tilde{\eta}(j)} &= \tau^{-1} \left(\overline{\begin{pmatrix} d & j \\ t^{-1}(\tilde{\eta}(j)) & \iota(d) \end{pmatrix}} \right) \\
&= \tau^{-1} \left(\begin{pmatrix} \iota(d) & j \\ t^{-1}(\tilde{\eta}(j)) & d \end{pmatrix} \right) = \frac{N(\lambda)}{\delta} \iota(d)u_1 + du_2 + j + \tilde{\eta}(j).
\end{aligned}$$

Since $\mathbf{1} = \frac{N(\lambda)}{\delta} u_1 + u_2$, it follows that each element in the K -vector subspace $\{k\mathbf{1} + j + \tilde{\eta}(j) \mid k \in K, j \in M_1\}$ is fixed by the involution. Since

$$\{k\mathbf{1} + j + \tilde{\eta}(j) \mid k \in K, j \in M_1\} \oplus Ks_0 = X,$$

we conclude that $\mathcal{H} = \{k\mathbf{1} + j + \tilde{\eta}(j) \mid k \in K, j \in M_1\}$.

By the definition of \star and $x \mapsto \overline{x}$, we have

$$V_{x, s_0 \star y} z = \tau^{-1}(V_{\tau(x), s'_0 \tau(y)} \tau(z)).$$

It follows from Theorem 3.17 that

$$\begin{aligned}
V_{x, s_0 \star y} z &= \tau^{-1} \left(\frac{1}{2}(\tau(x)\tau(y)\tau(z)) + \langle \tau(y), \tau(z) \rangle \tau(x) \right. \\
&\quad \left. + \langle \tau(y), \tau(x) \rangle \tau(z) + \langle \tau(x), \tau(z) \rangle \tau(y) \right).
\end{aligned}$$

Since τ is an isometry we have

$$\begin{aligned}
V_{x, s_0 \star y} z &= \frac{1}{2}(xyz + \langle y, z \rangle x + \langle y, x \rangle z + \langle x, z \rangle y) \\
&= \frac{1}{2} \left(\frac{1}{2}(x(h(y, z) + h(z, y)) + y(h(x, z) + h(z, x)) + z(h(x, y) + h(y, x))) \right. \\
&\quad \left. + g(y, z)x + g(y, x)z + g(x, z)y \right) \\
&= \frac{1}{2}(xh(y, z) + yh(x, z) + zh(y, x)),
\end{aligned}$$

where the last step follows from

$$\begin{aligned} h(y, z) + h(z, y) &= h(y, z) - \overline{h(y, z)} \\ &= 2h(y, z) - f(h(y, z), 1)1 = 2(h(y, z) - g(y, z)1). \end{aligned}$$

Again it follows from Theorem 3.17 and Theorem 4.1 that

$$\begin{aligned} (x \star \bar{y} - y \star \bar{x}) \star s_0 &= \tau^{-1} \left((\tau(x)\overline{\tau(y)} - \tau(y)\overline{\tau(x)}) s'_0 \right) \\ &= \tau^{-1} (\langle \tau(x), \tau(y) \rangle 1) \\ &= \langle x, y \rangle \mathbf{1} = g(x, y) \mathbf{1}; \\ \nu(x) &= \frac{1}{12\mu} \langle x, xxx \rangle = -\frac{\delta^2}{N(\lambda)^2} q(\pi(x)). \end{aligned}$$

Finally, if we make other choices for $a' \in X^*$ and $\lambda' \in \Delta$, then the structurable algebras $X(\Omega, a, \lambda)$ and $X(\Omega, a', \lambda')$ are, by construction, isometric as Freudenthal triple systems to the Freudenthal triple system X . It follows from Lemma 3.20 that they are isotopic. \square

Remark 5.5. In the case of quadrangular algebras of type E_6 , E_7 and E_8 , we can actually take $\lambda = 1$, in which case the formulas of Theorem 5.4 look nicer; see Lemma 7.5 below. We do not know whether we can always take $\lambda = 1$ in the pseudo-quadratic case.

Two quadrangular algebras are isotopic if and only if they describe the same Moufang quadrangle. For a precise definition and some properties, we refer to [W1, Chapter 8]. In the following lemma we observe that when we construct two structurable algebras starting from two isotopic quadrangular algebras, we end up with isotopic structurable algebras.

Lemma 5.6. *Let $\Omega = (K, L, q, 1, X, \cdot, h, \theta)$ and $\Omega' = (K, L', q', 1', X', \cdot', h', \theta')$ be two isotopic quadrangular algebras. Then the structurable algebras constructed from X and from X' as in the previous corollary are isotopic.*

Proof. By Lemma 3.20 it suffices to show that X and X' have isometric Freudenthal triple systems, since the Freudenthal triple systems of the quadrangular algebra are isometric to the Freudenthal triple systems of the obtained structurable algebras.

If Ω and Ω' are isotopic, then Ω' is isomorphic to the isotope Ω_u for some $u \in L$; we denote the corresponding isomorphisms from L_u to L' and from X_u to X' by α and ψ , respectively. We use the following formulas from [W1, Proposition 8.1]:

$$\begin{aligned} 1' &= \alpha(u), \\ \theta'(\psi(x), \alpha(v)) &= q(u)^{-1} \theta(x, v), \\ \psi(x) \cdot' \alpha(v) &= (xv)u^{-1}, \end{aligned}$$

for all $x \in X$ and all $v \in L$. It follows that

$$\begin{aligned}\psi(x) \cdot' \pi'(\psi(x)) &= \psi(x) \cdot' \theta'(\psi(x), 1') = q(u)^{-1}(x\theta(x, u))u^{-1} \\ &= q(u)^{-1}((x\pi(x))u)u^{-1} = q(u)^{-1}x\pi(x).\end{aligned}$$

By linearizing this expression we obtain that the Freudenthal triple systems are similar with isometry ψ and multiplier $q(u)^{-1}$. \square

Remark 5.7. We do not know whether the converse also holds, in other words, whether the fact that the structurable algebras are isotopic implies that the quadrangular algebras are isotopic.

The compatibility of the Freudenthal triple system with the $C(q, 1)$ -module structure of X translates into the following corollary, showing that the V -operators of the structurable algebra in Theorem 5.4 also behave well with respect to the $C(q, 1)$ -module structure of X .

Corollary 5.8. *For all $x, y, z \in X$ and $v \in L \setminus \{0\}$ we have that*

$$(V_{x, s_0 \star y} z)v = \frac{V_{xv, s_0 \star(yv)} zv}{q(v)}.$$

Proof. From Theorem 4.3 we have that $(xyz)v = \frac{(xv)(yv)(zv)}{q(v)}$, and from [W1, Proposition 4.18] it follows that $g(xv, yv) = g(x, y)q(v)$. Hence

$$\begin{aligned}V_{xv, s_0 \star(yv)} zv &= \frac{1}{2} \left((xv)(yv)(zv) + \langle(yv), (zv)\rangle xv \right. \\ &\quad \left. + \langle(yv), (xv)\rangle zv + \langle(xv), (zv)\rangle yv \right) \\ &= \frac{q(v)}{2} \left((xyz)v + \langle y, z \rangle xv + \langle y, x \rangle zv + \langle x, z \rangle yv \right) \\ &= q(v)(V_{x, s_0 \star y} z)v. \quad \square\end{aligned}$$

5.2 Inverses in quadrangular algebras and in structurable algebras

In this section we show that there is a relation between the conjugate inverse of an element in a structurable algebra and a map that behaves like an inverse in a quadrangular algebra.

In Theorem 3.19 the conjugate inverse of elements in structurable algebras of skew-dimension one is characterized. We apply this to the structurable algebra we obtained in Theorem 5.4.

Since the map ν is a scalar multiple of $q \circ \pi$ and $q \circ \pi$ is anisotropic, each element in X except 0 has a conjugate inverse, and hence the structurable

algebra X is a division algebra. For each $u \in X \setminus \{0\}$ the conjugate inverse is given by

$$\hat{u} = -\frac{1}{3\mu\nu(u)} s_0 \star \{u, s_0 \star u, u\} = s_0 \frac{u\pi(u)}{q(\pi(u))}. \quad (5.1)$$

In quadrangular algebras, we also encounter a certain expression that behaves like an inverse. This expression occurs in the calculation of the so-called μ -maps of a Moufang polygon; these μ -maps are certain automorphisms of order two of the Moufang polygon reflecting an apartment. We refer to [TW, Chapter 6] for the precise definition (which is not relevant for our purposes).

In [TW, Chapter 32], these μ -maps are calculated explicitly. We will illustrate with a few examples in which sense the μ -map behaves like an inverse.

Examples 5.9. (1) In the case of Moufang triangles, the root groups are parametrized by an alternative division algebra A ; see section 2.2.1. Then for every $t \in A \setminus \{0\}$, we have

$$\mu(x_1(t)) = x_4(t^{-1}) x_1(t) x_4(t^{-1}).$$

(2) In the case of Moufang hexagons, the root group U_1 is parametrized by an anisotropic cubic norm structure J ; see section 2.2.2. For each $a \in J \setminus \{0\}$, we define $a^{-1} := a^\# / N(a)$, and we have

$$\mu(x_1(a)) = x_7(a^{-1}) x_1(a) x_7(a^{-1}).$$

(3) In the case of Moufang quadrangles of quadratic form type, the root group U_4 is parametrized by a vector space V equipped with an anisotropic quadratic form q ; see section 2.2.4. For each $v \in V \setminus \{0\}$, we have

$$\mu(x_4(v)) = x_0(v/q(v)) x_4(v) x_0(v/q(v)).$$

Observe that the element $v/q(v)$ is precisely the inverse of v in the Clifford algebra $C(q)$.

(4) In our last example, we consider Moufang quadrangles arising from a quadrangular algebra $\Omega = (K, L, q, 1, X, \cdot, h, \theta)$. In this case, U_1 is parametrized by $X \times K$, equipped with a group operation

$$(a, s) \cdot (b, t) = (a + b, s + t + g(b, a))$$

for all $a, b \in X$ and all $s, t \in K$; see [W1, Chapter 11]. In the E_6 , E_7 and E_8 case, the corresponding μ -map is calculated in [TW, (32.8)], but it is easily verified that this formula holds for any quadrangular algebra. For all $x \in X \setminus \{0\}$, we have

$$\mu(x_1(x, 0)) = x_5\left(\frac{x\pi(x)}{q(\pi(x))}, 0\right) x_1(x, 0) x_5\left(\frac{x\pi(x)}{q(\pi(x))}, 0\right).$$

(The general formula for $\mu(x_1(x, s))$ is more involved, but similar in style.) Observe that the expression $\frac{x\pi(x)}{q(\pi(x))}$ is almost equal to the conjugate inverse of x ; see equation (5.1). The fact that the s_0 is missing is explained by the fact that in the expression

$$V_{u, \hat{u}} y = V_{u, s_0 \star \frac{u\pi(u)}{q(\pi(u))}} y = y,$$

the s_0 is necessary to translate the V -operator into the quadrangular algebra context; see Theorem 5.4.

6 Structurable algebras on pseudo-quadratic spaces

There is a standard procedure to construct a structurable algebra from a hermitian form; this is precisely case (3) from the classification we mentioned in Remark 3.16.

Theorem 6.1 ([A1, Examples 8.iii]). *Suppose $(\mathcal{E}, \bar{\cdot})$ is a unital associative algebra over a field K with involution $\bar{\cdot}$. Let \mathcal{W} be a unital left \mathcal{E} -module. Suppose $h : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{E}$ is a hermitian form, i.e.*

- *h is bilinear over K and $h(ew_1, w_2) = eh(w_1, w_2)$,*
- *$\overline{h(w_1, w_2)} = h(w_2, w_1)$,*

for all $e \in \mathcal{E}, w_1, w_2 \in \mathcal{W}$. Then $\mathcal{E} \oplus \mathcal{W}$ is a structurable algebra with the following involution and multiplication:

$$\begin{aligned} \overline{e + w} &= \bar{e} + w, \\ (e_1 + w_1)(e_2 + w_2) &= (e_1 e_2 + h(w_2, w_1)) + (e_2 w_1 + \bar{e}_1 w_2), \end{aligned}$$

for all $e, e_1, e_2 \in \mathcal{E}$ and all $w, w_1, w_2 \in \mathcal{W}$.

When we start with a pseudo-quadratic space, we have a skew-hermitian form at our disposal; see Definition 2.3 above. In characteristic different from two there is a standard procedure to make a skew-hermitian form into a hermitian form.

We show that when we use this method to construct a structurable algebra on a pseudo-quadratic space defined over a *quadratic pair*, we get an algebra that is isotopic to the family of isotopic algebras constructed in Theorem 5.4.

Definition 6.2 ([W1, Definition 1.12]). Let L be a skew field and σ an involution of L . We call (L, σ) a *quadratic pair*⁵, if either

⁵This notion, taken from [W1, Definition 1.12], is quite different from the notion of a quadratic pair as defined in the Book of Involutions [KMRT], and has nothing to do with the notion of a quadratic pair in (finite) group theory either.

- (i) L/K is a separable quadratic field extension and σ is the generator of the Galois group; or
- (ii) L is a quaternion algebra over K and σ is the standard involution.

Define $q(u) = uu^\sigma$; then $(K, L, q, 1)$ is a pointed anisotropic quadratic space.

A pseudo-quadratic space (L, σ, X, h, π) where (L, σ) is a quadratic pair, is called *standard* if

$$\pi(xu) = u^\sigma \pi(x)u$$

for all $x \in X$ and all $u \in L$. In [W1, Proposition 1.18] it is shown that a standard anisotropic pseudo-quadratic space over a quadratic pair is a quadrangular algebra.

The following definition is used in the E_6 , E_7 and E_8 case in [TW].

Definition 6.3. Let (L, σ, X, h, π) be a pseudo-quadratic space in characteristic not 2. We fix an arbitrary element $\xi \in X$; then ξL becomes a subspace of X . We define the *orthogonal complement* of ξL as

$$(\xi L)^\perp = \{x \in X \mid g(x, \xi v) = 0 \text{ for all } v \in L\}.$$

Lemma 6.4. Let (L, σ, X, h, π) be a pseudo-quadratic space in characteristic not 2. Then $X = \xi L \oplus (\xi L)^\perp$. Moreover, for every $x \in X$, we have that $x \in (\xi L)^\perp$ if and only if $h(x, \xi) = 0$.

Proof. Our proof is essentially as in [TW, (13.51)]. The first assertion follows since g is nondegenerate. To prove the second assertion let $x \in X$, and observe that for each $v \in L$, we have

$$g(x, \xi v) = \frac{1}{2}f(h(x, \xi v), 1) = \frac{1}{2}f(h(x, \xi), v).$$

The claim now follows since both f and g are nondegenerate. \square

The quadratic pair (L, σ) is a unital associative algebra with involution, but $(\xi L)^\perp$ is a right L -module equipped with a skew-hermitian form. In the following lemma, inspired by [TW, (16.18)], we redefine the involution on L , the scalar multiplication on $(\xi L)^\perp$ and the skew-hermitian form, in such a way that we get a module that satisfies the requirements of Theorem 6.1.

We embed L in X by considering ξL , which we view as a unital associative algebra with involution by defining

$$(\xi v)(\xi w) = \xi vw \quad \text{and} \quad (\xi v)^\sigma = \xi v^\sigma,$$

for all $v, w \in L$.

Lemma 6.5. *Let (L, σ, X, h, π) be a pseudo-quadratic space, and take an element $e \in L$ such that $e^\sigma = -e$. Then the map $v \mapsto \bar{v} = ev^\sigma e^{-1}$ is an involution of L , and $\{s \in L \mid \bar{s} = -s\} = Ke$.*

For each $v \in L$ and each $x \in (\xi L)^\perp$, we define $(\xi v) \circ x := x\bar{v}$. Now let

$$h': (\xi L)^\perp \times (\xi L)^\perp \rightarrow \xi L: (x, y) \mapsto h'(x, y) := \xi eh(x, y).$$

Then $(\xi L)^\perp$ is a left ξL -module w.r.t. \circ , and h' is a hermitian form on $(\xi L)^\perp$ w.r.t. the involution $\bar{}$, satisfying the requirements of Theorem 6.1.

Proof. It is clear that the map $v \mapsto \bar{v}$ is an involution. In order to determine the subspace of skew-symmetric elements $\mathcal{S} = \{s \in L \mid \bar{s} = -s\}$, we first observe that $\bar{e} = ee^\sigma e^{-1} = -e$, hence $e \in \mathcal{S}$. On the other hand, for each $s \in L$, we have

$$\begin{aligned} \bar{s} = -s &\iff se = -es^\sigma \\ &\iff se = (se)^\sigma \\ &\iff se \in \text{Fix}_L(\sigma) = K. \end{aligned}$$

It follows that $\dim_K \mathcal{S} = 1$; since $e \in \mathcal{S}$, we conclude that $\mathcal{S} = Ke$.

Let $v, w \in L$ and $x \in (\xi L)^\perp$, since $h(x\bar{v}, \xi) = vh(x, \xi) = 0$ we have that $(\xi v) \circ x \in (\xi L)^\perp$. To verify that $(\xi L)^\perp$ is a left ξL -module, it is enough to check the following

$$\xi v \circ (\xi w \circ x) = (x\bar{w})\bar{v} = x(\bar{w} \bar{v}) = x(\overline{vw}) = (\xi vw) \circ x.$$

Finally, it is easily checked that h' is a hermitian form:

$$\begin{aligned} h'(\xi v \circ x, y) &= \xi eh(x\bar{v}, y) = \xi e(ev^\sigma e^{-1})^\sigma h(x, y) = (\xi v)h'(x, y), \\ \overline{h'(x, y)} &= \overline{-h(x, y)e} = -eh(x, y)^\sigma = h'(y, x), \end{aligned}$$

for all $x, y \in (\xi L)^\perp$ and all $v \in L$. □

Theorem 6.6. *Let (L, σ, X, h, π) be a pseudo-quadratic space, and choose an element $e \in L$ such that $e^\sigma = -e$. Since $X = (\xi L) \oplus (\xi L)^\perp$, each element in X can be written in a unique way as $\xi v + x$ for $v \in L$ and $x \in (\xi L)^\perp$.*

Then X is a structurable algebra, with involution and multiplication given by

$$\begin{aligned} \overline{\xi v + x} &= \xi(ev^\sigma e^{-1}) + x, \\ (\xi v + x) \cdot (\xi u + y) &= \xi(vu + eh(y, x)) + (x(eu^\sigma e^{-1}) + yv), \end{aligned} \tag{6.1}$$

for all $u, v \in L$ and all $x, y \in (\xi L)^\perp$. This structurable algebra has skew-dimension one.

When we fix $e = h(\xi, \xi)$, then this algebra is isotopic to the family of structurable algebras obtained in Theorem 5.4 starting from the pseudo-quadratic space (L, σ, X, h, π) .

In the corresponding Freudenthal triple system we have $yyy = 6y\pi(y)$ for all $y \in X$.

Proof. We consider $(\xi L)^\perp$ as a left hermitian space over ξL with involution and hermitian form as in Lemma 6.5. Writing down the involution and multiplication in Theorem 6.1 gives the above formulas.

Since $(\xi L)^\perp$ is invariant under the involution, it follows from Lemma 6.5 that this structurable algebra has skew-dimension one. We will denote this structurable algebra, obtained from the hermitian form, by X .

On the other hand, let \tilde{X} be the structurable algebra obtained from the quadrangular algebra (L, σ, X, h, π) , as in Theorem 5.4. We will show that X and \tilde{X} are isotopic by showing that their associated Freudenthal triple systems are similar.

Now take $e = h(\xi, \xi) \in L$; we have $\bar{e} = -e$. We will determine the trilinear map of the corresponding Freudenthal triple system of X (see Theorem 3.17) for $s_0 = e$. Let $y = \xi v + x \in (\xi L) \oplus (\xi L)^\perp$ be arbitrary. Then

$$2V_{y, s_0 y} y = 2(2(y \cdot \overline{s_0 \cdot y})y - (y \cdot \bar{y}) \cdot (s_0 \cdot y)) \quad (6.2)$$

$$= 6(\xi(-vev^\sigma + eh(x, x)e)v + x(e(vev^\sigma e^{-1} - eh(x, x))) \quad (6.3)$$

$$= 12(\xi(-\pi(\xi v^\sigma + xe)v) + x(e\pi(\xi v^\sigma + xe)e^{-1})); \quad (6.4)$$

equation (6.2) follows from the definition of the V -operator of a structurable algebra; (6.3) follows after a straightforward calculation using the multiplication on X defined by (6.1); and (6.4) follows from

$$\begin{aligned} vev^\sigma - eh(x, x)e &= vh(\xi, \xi)v^\sigma - eh(x, x)e = h(\xi v^\sigma, \xi v^\sigma) + h(xe, xe) \\ &= h(\xi v^\sigma + xe, \xi v^\sigma + xe) = 2\pi(\xi v^\sigma + xe), \end{aligned}$$

since $xe \in (\xi L)^\perp$.

It follows from Theorem 5.4 that in \tilde{X} the trilinear map of the corresponding Freudenthal triple system is $2V_{\tilde{y}, s_0 \tilde{y}} \tilde{y} = 3\tilde{y}h(\tilde{y}, \tilde{y}) = 6\tilde{y}\pi(\tilde{y})$ for all $\tilde{y} \in \tilde{X}$.

Let ψ be the vector space isomorphism

$$\psi: \tilde{X} \rightarrow X: \xi v + x \mapsto \xi v^\sigma + xe^{-1};$$

then by applying (6.4) with y replaced by $\psi(y) = \xi v^\sigma + x e^{-1}$, we get

$$\begin{aligned} 2V_{\psi(y), s_0\psi(y)} \psi(y) &= 12(-\xi(\pi(y)v^\sigma) + x\pi(y)e^{-1}) \\ &= 12\psi(\xi(v\pi(y)) + x\pi(y)) \\ &= 12\psi(y\pi(y)) \\ &= 2\psi(6y\pi(y)). \end{aligned}$$

The expression $V_{\psi(y), s_0\psi(y)} \psi(y)$ is exactly the triple product $\psi(y)\psi(y)\psi(y)$ in the Freudenthal triple system of the structurable algebra X , whereas the expression $6y\pi(y)$ is the triple product yyy in the Freudenthal triple system of the structurable algebra \tilde{X} .

We conclude that ψ is a similarity of Freudenthal triple systems, and therefore the structurable algebras X and \tilde{X} are isotopic (see Lemma 3.20). \square

7 Structurable algebras on quadrangular algebras of type E_6 , E_7 and E_8

In this section we consider quadrangular algebras of type E_6 , E_7 and E_8 . We investigate the structurable algebra obtained in Theorem 5.4 for a “nice” choice of a and e .

In the E_8 -case, we show that this structurable algebra is a twisted version of the Jordan algebra of a biquaternion algebra over the base field K . This algebra can also be obtained by considering a generalized Cayley–Dickson process, see [AF1], starting from the Jordan algebra of a biquaternion algebra over K . A similar description holds for E_6 and E_7 . More precisely, we will show the following result.

Theorem 7.1. *Let Ω be a quadrangular algebra of type E_6 , E_7 or E_8 over K , with $\text{char}(K) \neq 2$, and let X be the structurable algebra obtained from Ω as in Theorem 5.4 (which is uniquely defined up to isotopy). Then X is isotopic to $\text{CD}(A^+, \text{Nrd}, c)$ for some division algebra A and some $c \in K$, where*

- (i) A is a quaternion algebra Q if Ω is of type E_6 ;
- (ii) A is a tensor product $Q \otimes L$ with Q a quaternion algebra and L a quadratic extension, if Ω is of type E_7 ;
- (iii) A is a biquaternion algebra $Q_1 \otimes Q_2$ if Ω is of type E_8 .

We will give more precise statements below; in particular, we will explicitly construct the algebra A and the constant c in each case.

7.1 Coordinatization

For an explicit description of the quadrangular algebra of type E_6 , E_7 and E_8 , we refer to [TW, Chapter 12 and 13]; for a concise description we refer to the first part of [W1, Chapter 10]. Some care is needed, since the map g in [TW] is equal to $-g$ in [W1]. Here we only give a concise overview of the structure of a quadrangular algebra of type E_6 , E_7 or E_8 .

Definition 7.2. A quadratic space (K, L, q) with base point is of type E_6 , E_7 or E_8 if it is anisotropic and there exists a separable quadratic field extension E/K , with norm denoted by N , such that:

E_6 : there exist $s_2, s_3 \in K^*$ such that

$$(K, L, q) \cong (K, E^3, N \perp s_2N \perp s_3N);$$

E_7 : there exist $s_2, s_3, s_4 \in K^*$ such that $s_2s_3s_4 \notin N(E)$ and

$$(K, L, q) \cong (K, E^4, N \perp s_2N \perp s_3N \perp s_4N);$$

E_8 : there exist $s_2, s_3, s_4, s_5, s_6 \in K^*$ such that $-s_2s_3s_4s_5s_6 \in N(E)$ and

$$(K, L, q) \cong (K, E^6, N \perp s_2N \perp s_3N \perp s_4N \perp s_5N \perp s_6N).$$

We always assume that $s_2s_3s_4s_5s_6 = -1$, which can be achieved by rescaling the quadratic form if necessary. We use the convention that $s_{ij} = s_i s_j$ and $s_{ijk} = s_i s_j s_k$ for all $i, j, k \in \{2, \dots, 6\}$.

As we are working in characteristic not 2, we will always assume that $E = K(\gamma)$ with $\gamma^2 \in K$.

It is shown in [TW, (12.37)] that if

$$(K, E^6, N \perp s_2N \perp s_3N \perp s_4N \perp s_5N \perp s_6N)$$

is a quadratic space of type E_8 , then $(K, E^4, N \perp s_2N \perp s_3N \perp s_4N)$ is a quadratic space of type E_7 and $(K, E^3, N \perp s_2N \perp s_3N)$ is a quadratic space of type E_6 .

If (K, L, q) is a quadratic space with base point of type E_8 , L has dimension 12 over K and there exists a scalar multiplication $E \times L \rightarrow L$ that extends the scalar multiplication $K \times L \rightarrow L$. We denote a basis of L by $(v_1, \gamma v_1, \dots, v_6, \gamma v_6)$; with this notation v_1 is the base point of q .

Let $(K, L, q, 1, X, \cdot, h, \theta)$ be a quadrangular algebra of type E_6 , E_7 or E_8 , then (K, L, q) is a quadratic space of type E_6 , E_7 or E_8 with basepoint denoted by 1.

The vector space X has K -dimension equal to 8, 16 or 32, respectively; it is a $C(q, 1)$ -module. Some of the properties of the maps \cdot, h, θ and π are

given in Definition 3.1. The existence of the vector space X and of the maps \cdot , h and θ is shown in [TW, Chapter 13] by giving an explicit ad-hoc construction using the coordinatization of L .

Let $\mathcal{I} = \{2, 3, 4, 5, 6, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56\}$; in the E_8 -case, an arbitrary element $x \in X$ can be written as $x = t_1 v_1 + \sum_{i \in \mathcal{I}} t_i v_i$ with coefficients $t_i \in E$. The quadrangular algebra in the E_7 -case is a subspace of the E_8 -case by taking all the elements in X for which $t_5 = t_6 = t_{25} = t_{26} = t_{35} = t_{36} = t_{45} = t_{46} = 0$. The E_6 -case is the subspace of the E_7 -case consisting of the elements for which $t_4 = t_{24} = t_{34} = t_{56} = 0$.

Since the E_8 -case is the largest and least understood, we focus primarily on that case. Most of our results in the E_6 - and E_7 -case can then be deduced by making the appropriate coefficients zero.

We start by investigating the set

$$X|_K := \left\{ t_1 v_1 + \sum_{i \in \mathcal{I}} t_i v_i \mid t_1, \dots, t_{56} \in K \right\} \leq X.$$

This is a 16-dimensional vector space over K . Let

$$L|_K = \{ t_1 v_1 + t_2 v_2 + t_3 v_3 + t_4 v_4 + t_5 v_5 + t_6 v_6 \mid t_1, \dots, t_6 \in K \} \leq L,$$

and denote the restriction of the quadratic form q to $L|_K$ by $q_K: L|_K \rightarrow K$. By construction, see [TW, (13.5) and (13.8)], $X|_K$ is isomorphic as a vector space to $C(q_K, 1)/M_K$, where M_K is the submodule $(v_2 v_3 v_4 v_5 v_6 - 1)C(q_K, 1)$ of $C(q_K, 1)$.

Since $v_i v_j = -v_j v_i \in C(q_K, 1)$ for $i \neq j \in \{2, \dots, 5\}$, the element $v_2 v_3 v_4 v_5 v_6$ is in the center of $C(q_K, 1)$; therefore M_K is a two-sided ideal of $C(q_K, 1)$.

Lemma 7.3. *Consider $X|_K$ as an associative algebra endowed with the multiplication induced by the Clifford algebra with base point. Then $X|_K$ is isomorphic, as an algebra, to a biquaternion algebra.*

Proof. The multiplication on $X|_K = C(q_K, 1)/M_K$ is induced by the multiplication in the Clifford algebra $C(q_K, 1)$ by reducing the result modulo M_K ; see [TW, (13.8)].

We define two quaternion algebras over K with the following generators:

$$\begin{aligned} Q_1 &:= (-s_2, -s_3)_K = \langle \ell, m \mid \ell^2 = -s_2, m^2 = -s_3, \ell m = -m \ell \rangle, \\ Q_2 &:= (-s_{46}, -s_{56})_K = \langle n, r \mid n^2 = -s_{46}, r^2 = -s_{56}, nr = -nr \rangle. \end{aligned}$$

In order to construct an isomorphism $\psi: Q_1 \otimes_K Q_2 \rightarrow X|_K$, we have to describe two isomorphisms $\psi_i: Q_i \rightarrow X|_K$ for $i \in \{1, 2\}$, such that the images

$\psi_1(Q_1)$ and $\psi_2(Q_2)$ commute elementwise, and together generate $X|_K$. We can achieve this by the choice

$$\begin{aligned}\psi_1(\ell) &= v_2, & \psi_1(m) &= v_3, & \psi_1(\ell m) &= v_{23}, \\ \psi_2(n) &= v_{46}, & \psi_2(r) &= v_{56}, & \psi_2(nr) &= s_6 v_{45}.\end{aligned}$$

Observe that the subspaces $\langle 1, v_2, v_3, v_{23} \rangle$ and $\langle 1, v_{46}, v_{56}, v_{45} \rangle$ do indeed commute elementwise, and together they generate $X|_K$. \square

We can summarize the isomorphism ψ in the following table:

\otimes	1	n	r	nr
1	v_1	v_{46}	v_{56}	$s_6 v_{45}$
ℓ	v_2	$-s_{246} v_{35}$	$s_{256} v_{34}$	$s_{2456} v_{36}$
m	v_3	$s_{346} v_{25}$	$-s_{356} v_{24}$	$-s_{3456} v_{26}$
ℓm	v_{23}	$-s_{2346} v_5$	$s_{2356} v_4$	$-v_6$

- Remark 7.4.** (i) The construction of the biquaternion algebra depends on the similarity class of q and on the norm splitting for q , and in fact, this biquaternion algebra is *not* an invariant of the quadrangular algebra.
- (ii) The Albert form corresponding to the biquaternion algebra is similar to the 6-dimensional quadratic form $\langle 1, s_2, s_3, s_4, s_5, s_6 \rangle$. Since this quadratic form is anisotropic, we see in particular that the biquaternion algebra $Q_1 \otimes Q_2$ is always a division algebra.
- (iii) In the E_6 -case $X|_K$ is isomorphic to Q_1 , and in the E_7 -case $X|_K$ is isomorphic to $Q_1 \otimes_K K(r)$.

The following rather technical lemma will assure that we can choose the structurable algebra X obtained in Theorem 5.4 in a nice way.

Lemma 7.5. *When applying Theorem 5.4 with $a = -v_1/\gamma$, we can always choose e in such a way that $\lambda = 1$. For those choices we have that $\mathbf{1} = v_1, s_0 = -\gamma v_1$ and $\mu = \gamma^2$.*

Proof. Recall that $\gamma \in E \setminus K$ and $\gamma^2 \in K$. For $a = -v_1/\gamma$ we have that $q(\pi(a)) = -\frac{1}{\gamma^2}$ and $a\pi(a) = \frac{v_1}{\gamma^2}$. So $\delta = \frac{1}{\gamma}$, and hence $\Delta = K(\delta) = E$. We point out that in $X \otimes_K \Delta$, the element $1 \otimes \delta$ is not equal to $\frac{1}{\gamma} \otimes 1$. For instance, we have

$$\frac{a\pi(a)}{\delta} = \frac{v_1}{\gamma^2} \otimes \gamma \neq \frac{v_1}{\gamma} \otimes 1 = -a.$$

If we assume that $\lambda = 1$ using the formulas in Theorem 5.4, we obtain $\mathbf{1} = v_1, s_0 = -\gamma v_1, \mu = \gamma^2$. In order to prove that we can always find an e such that $\lambda = 1$, we will do some explicit calculations.

We have

$$u'_1 = \frac{1}{2} \left(-\frac{v_1}{\gamma} \otimes 1 + v_1 \otimes \frac{1}{\gamma} \right), \quad u'_2 = \frac{1}{2} \left(\frac{v_1}{\gamma} \otimes \gamma + v_1 \otimes 1 \right).$$

We determine explicitly the subspaces M_1 and M_{-1} . One can calculate that for all $x = \sum_{I \in \mathcal{I}} t_I v_I \in X$,

$$u'_1 u'_2(x \otimes 1) = x/\gamma \otimes \gamma, \quad u'_1 u'_2(x \otimes \gamma) = \gamma x \otimes 1,$$

and for all $x = t_1 v_1 \in X$,

$$u'_1 u'_2(x \otimes 1) = 2(x/\gamma \otimes \gamma), \quad u'_1 u'_2(x \otimes \gamma) = 2(\gamma x \otimes 1).$$

We find that

$$\begin{aligned} M_1 &= \left\{ \gamma x \otimes 1 + x \otimes \gamma \mid x = \sum_{I \in \mathcal{I}} t_I v_I \text{ with } t_I \in E \right\}, \\ M_{-1} &= \left\{ \gamma x \otimes 1 - x \otimes \gamma \mid x = \sum_{I \in \mathcal{I}} t_I v_I \text{ with } t_I \in E \right\}. \end{aligned} \tag{7.1}$$

Following an idea of Richard Weiss, we introduce the following notation: let i, j, k, l, m denote five different indices in $\{2, \dots, 6\}$, then $\beta_{ijkl} = \pm 1$ is defined by $v_{ij} v_{kl} = \beta_{ijkl} s_i s_j s_k s_l v_m$.

Next we need an expression for $g(u'_1, e\pi(e))$ for an arbitrary $e \in M_1$. (Recall that g is now a map from $(X \otimes_K \Delta) \times (X \otimes_K \Delta)$ to Δ .)

So let $x = \sum_{2 \leq i \leq 6} t_i v_i + \sum_{2 \leq i < j \leq 6} t_{ij} v_{ij} \in X$ for $t_i, t_{ij} \in E$, and consider the expression

$$\rho(x) := \sum_{ij/kl/m} \beta_{ijkl} t_m t_{ij} t_{kl} \in E = \Delta,$$

where the summation runs over all partitions of $\{2, \dots, 6\}$ into two sets of two elements and one set of one element.

Since in the E_6 -case no such partition with non-zero coefficients exists, $\rho(x)$ is identically 0 in this case.

Take $e = \gamma x \otimes 1 + x \otimes \gamma \in M_1$; then it follows from a lengthy computation⁶ that

$$g(u'_1, e\pi(e)) = 16\gamma^4 \rho(x) \in \Delta.$$

In the E_6 -case, this expression is identically 0, so $\lambda = 1$ by definition. In the E_7 - and the E_8 -case, we look for an $e \in M_1$ such that $g(u'_1, e\pi(e)) = 2$, so $8\gamma^4 \rho(x)$ should be equal to 1. This is indeed the case for

$$x = \frac{1}{2\gamma^2} (v_2 + \gamma v_{34} + \gamma v_{56}),$$

since $\beta_{3456} = 1$. □

⁶We wrote a computer program in Sage [Sage] to perform this computation for us.

We now show that, with these choices of a and e , the structurable algebra X is a “twisted” Jordan algebra of a biquaternion algebra, i.e. when we restrict the coefficients (w.r.t. the standard basis) to K , the algebra X is a Jordan algebra of a biquaternion algebra, but when we allow coefficients of E , we have to apply the non-trivial Galois automorphism σ of E/K at various (compatible) places.

Theorem 7.6. *Let \circ denote the multiplication of*

$$(X|_K)^+ \cong ((-s_2, -s_3)_K \otimes_K (-s_{46}, -s_{56})_K)^+,$$

the Jordan algebra obtained⁷ from the (associative) biquaternion algebra.

We choose $a = -v_1/\gamma$, and we let e be as in Lemma 7.5, such that $\lambda = 1$. Then the multiplication of the structurable algebra X obtained in Theorem 5.4, which we will denote by \star , is given by

$$\begin{aligned} Av_1 \star Bv_1 &= ABv_1 \\ Av_1 \star Bv_I &= A^\sigma Bv_I \\ Av_I \star Bv_1 &= ABv_I \\ Av_I \star Bv_I &= A^\sigma B(v_I \circ v_I) = (-s_I)A^\sigma Bv_1 \\ Av_I \star Bv_J &= A^\sigma B^\sigma(v_I \circ v_J) \end{aligned}$$

for all $A, B \in E$ and all $I \neq J \in \mathcal{I}$. The involution of X is given by

$$\overline{Av_1} = A^\sigma v_1, \quad \overline{Av_I} = Av_I$$

for all $A \in E$ and all $I \in \mathcal{I}$.

Proof. By Lemma 7.5, we have $\mathbf{1} = v_1$, $s_0 = -\gamma v_1$ and $\mu = \gamma^2$. We know that $s_0 \star s_0 = \mu \mathbf{1}$, so $\gamma v_1 \star \gamma v_1 = \gamma^2 v_1$. Since v_1 is the identity of \star , we have $Av_1 \star Bv_1 = ABv_1$ for all $A, B \in E$.

Since X has skew-dimension one, it follows from the definition of s_0 that

$$\mathcal{S} = \{x \in X \mid \bar{x} = -x\} = Ks_0 = K\gamma v_1.$$

From Theorem 5.4 we have

$$\mathcal{H} = \{x \in X \mid \bar{x} = x\} = \{k\mathbf{1} + j + \tilde{\eta}(j) \mid k \in K, j \in M_1\}.$$

It follows from equation (7.1) that every $j \in M_1$ is of the form $\gamma x \otimes 1 + x \otimes \gamma$ for some $x \in \bigoplus_{I \in \mathcal{I}} Ev_I$. Then $\tilde{\eta}(j) = \gamma x \otimes 1 - x \otimes \gamma \in M_{-1}$. It follows that

$$\{j + \tilde{\eta}(j) \mid j \in M_1\} = \{2\gamma x \otimes 1 \mid x \in \bigoplus_{I \in \mathcal{I}} Ev_I\} = \bigoplus_{I \in \mathcal{I}} Ev_I.$$

⁷If A is an associative algebra, then $x \circ y := \frac{1}{2}(xy + yx)$ makes A into a Jordan algebra. This Jordan algebra is denoted by A^+ .

Therefore

$$\mathcal{H} = Kv_1 \oplus \bigoplus_{I \in \mathcal{I}} Ev_I.$$

It follows that we have for $A \in E$ that

$$\overline{Av_1} = A^\sigma v_1, \quad \overline{Av_I} = Av_I \text{ for all } I \in \mathcal{I}.$$

For all $x \in \mathcal{H}$ and $y \in X$, we have

$$V_{x,1}y = (x \star \overline{1}) \star y + (y \star \overline{1}) \star x - (y \star \overline{x}) \star 1 = x \star y.$$

It now follows from Theorem 5.4 that

$$x \star y = V_{x, s_0 \star \frac{1}{\mu} s_0} y = -\frac{1}{2\gamma^2} \left(xh(\gamma v_1, y) + (\gamma v_1)h(x, y) + yh(x, \gamma v_1) \right). \quad (7.2)$$

Now we can compute $x \star y$ for all different values that can occur, using the formulas from [TW, (13.6) and (13.19)]. Let i, j, k, l be distinct indices in $\{2, 3, 4, 5, 6\}$; then one can verify the following multiplication table:

x	y	$x \star y$
Av_i	Bv_1	ABv_i
Av_i	Bv_i	$-s_i A^\sigma Bv_1$
Av_i	Bv_k	0
Av_i	Bv_{ik}	0
Av_i	Bv_{kl}	$2A^\sigma B^\sigma v_i v_{kl}$
Av_{ij}	Bv_1	ABv_{ij}
Av_{ij}	Bv_{ij}	$-s_i s_j A^\sigma Bv_1$
Av_{ij}	Bv_{ik}	0
Av_{ij}	Bv_{kl}	$2A^\sigma B^\sigma v_{ij} v_{kl}$

Observe that this multiplication coincides with the Jordan multiplication in $(X|_K)^+$ if A, B are restricted to K .

Note that the formula (7.2) is not valid for $x = \gamma v_1 \in \mathcal{S}$; this case is obtained by

$$Av_1 \star Bv_I = \overline{\overline{Bv_I} \star \overline{Av_1}} = A^\sigma Bv_I. \quad \square$$

Remark 7.7. The structurable algebra described above, consisting of a twisted Jordan algebra of a biquaternion algebra, is defined up to isotopy by the quadrangular algebra; in particular it is determined by the quadratic form of type E_6 , E_7 or E_8 . In the E_8 case, there is a strong relation between the Arason invariant of the quadratic form q , and the twisted Jordan algebra of the biquaternion algebra.

The Albert form of the biquaternion algebra in Theorem 7.6 is

$$q_A = \langle s_2, s_3, s_{23}, -s_{46}, -s_{56}, -s_{45} \rangle;$$

q_A is Witt equivalent to $\langle\langle -s_2, -s_3 \rangle\rangle \perp -\langle\langle -s_{46}, -s_{56} \rangle\rangle$. In fact, q_A is similar to $\langle 1, s_2, s_3, s_4, s_5, s_6 \rangle$ (note that $s_2 s_3 s_4 s_5 s_6 = -1$):

$$\begin{aligned} \langle s_2, s_3, s_{23}, -s_{46}, -s_{56}, -s_{45} \rangle &\simeq \langle s_2, s_3, s_{23}, s_{235}, s_{234}, s_{236} \rangle \\ &\simeq s_{23} \langle s_3, s_2, 1, s_5, s_4, s_6 \rangle \\ &\simeq s_{23} \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle. \end{aligned}$$

Let $I^3 K$ be the ideal in the Witt ring of K that is generated by the 3-fold Pfister forms over K ; this ideal consists precisely of the classes $[q]$ of quadratic forms q having even dimension, trivial discriminant, and trivial Hasse–Witt invariant.

The Arason invariant e_3 is a cohomological invariant⁸

$$\begin{aligned} e_3: \{3\text{-Pfister forms over } K\} &\rightarrow H^3(K, \mathbb{Z}/2\mathbb{Z}): \\ \langle\langle a, b, c \rangle\rangle &\mapsto (a) \cup (b) \cup (c); \end{aligned}$$

it extends to a well-defined group morphism $e_3: I^3 K \rightarrow H^3(K, \mathbb{Z}/2\mathbb{Z})$, that only depends on the similarity class of the quadratic form in the Witt ring (see [Ar]). It follows from the Milnor conjecture (see [V, 7.5]) that this induces an isomorphism

$$e_3: I^3 K / I^4 K \xrightarrow{\sim} H^3(K, \mathbb{Z}/2\mathbb{Z}).$$

Let q be a form of type E_8 ; then q has even dimension, trivial discriminant, and trivial Hasse–Witt invariant, so $q \in I^3 K$. In order to determine $e_3(q)$, we first rewrite q in the Witt ring (note that $N = \langle 1, -\gamma^2 \rangle$):

$$\begin{aligned} q &= N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle \\ &\simeq N \otimes (\langle\langle -s_2, -s_3 \rangle\rangle \perp -\langle\langle -s_{46}, -s_{56} \rangle\rangle) \\ &\simeq \langle\langle \gamma^2, -s_2, -s_3 \rangle\rangle \perp -\langle\langle \gamma^2, -s_{46}, -s_{56} \rangle\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} e_3(q) &= (\gamma^2) \cup (-s_2) \cup (-s_3) - (\gamma^2) \cup (-s_{46}) \cup (-s_{56}) \\ &= d(N) \cup ((-s_2) \cup (-s_3) - (-s_{46}) \cup (-s_{56})) \\ &= d(N) \cup c(\langle s_2, s_3, s_{23}, -s_{46}, -s_{56}, -s_{45} \rangle), \end{aligned}$$

where d denotes the image of the discriminant in $H^1(K, \mathbb{Z}/2\mathbb{Z})$, and c denotes the Hasse–Witt invariant in $H^2(K, \mathbb{Z}/2\mathbb{Z})$.

This invariant determines quadratic forms of type E_8 up to similarity: if q and q' are two quadratic forms of type E_8 with $e_3(q) = e_3(q') \in$

⁸The expression $(a) \cup (b) \cup (c)$ is the cup product of elements in $H^1(K, \mathbb{Z}/2\mathbb{Z})$.

$H^3(K, \mathbb{Z}/2\mathbb{Z})$, then $q \equiv q' \in I^4 K$, and from [H, Conjecture 1] for $k = 1$, it follows that q and q' are similar.

We conclude that the invariant $e_3(q)$ completely determines the isotopy class of the structurable algebra in Theorem 7.6. (On the other hand, notice that neither E nor the biquaternion algebra $(X|_K)^+$ are invariants of the quadratic form q or of the corresponding structurable algebra.)

Remark 7.8. The map $\nu = -1/\gamma^2 \cdot q \circ \pi$ is the conjugate norm on a structurable algebra of skew-dimension one. When we consider only the K -part of X , we obtain a nice expression for $q \circ \pi$ restricted to $X|_K$.

Indeed, in [AF2, Example 4.2] it is shown that the conjugate norm of a Jordan algebra is just its generic norm. When a Jordan algebra is obtained from a central simple associative algebra, its generic norm is equal to the reduced norm of the central simple algebra.

As $X|_K$ is the Jordan algebra arising from a biquaternion algebra, we find that

$$q(\pi(x)) = -\gamma^2 \nu(x) = N(\gamma) \text{Nrd}(z)$$

for $x \in X|_K$, where $x = \psi(z)$ for $z \in Q_1 \otimes Q_2$. This fact also follows from [AF1, Proposition 6.7], using the Cayley–Dickson process that we will explain in the next section.

7.2 The Cayley–Dickson process for structurable algebras

In [AF1, Section 6], Allison and Faulkner introduce a construction of structurable algebras starting from a certain class of Jordan algebras; this procedure is a generalization of the classical Cayley–Dickson doubling process. In the E_8 -case, the algebra described in Theorem 7.6 can also be obtained by applying this Cayley–Dickson process to the Jordan algebra obtained from a biquaternion algebra, as we will now explain.

In order to obtain a structurable algebra, one needs to start from a Jordan algebra equipped with a Jordan norm of degree 4. We will briefly explain the Cayley–Dickson process, and refer to [AF1] for more details.

Definition 7.9. Let J be a Jordan algebra over K . A form $Q: J \rightarrow K$ is a *Jordan norm of degree 4* if the following properties hold, for all $k \in K$ and all $j, j' \in J$:

- (i) $Q(kj) = k^4 Q(j)$;
- (ii) $Q(1) = 1$;
- (iii) $Q(U_j j') = Q(j)^2 Q(j')$;
- (iv) The trace form

$$T: J \times J \rightarrow K: (j, j') \mapsto Q(1; j)Q(1; j') - Q(1; j, j')$$

is a bilinear non-degenerate form.

Let Q be a Jordan norm of degree 4, and consider the linear automorphism θ on J given by

$$b^\theta = -b + \frac{1}{2}T(b, 1)1;$$

observe that $\theta^2 = 1$. Let $\mu \in K$. Then $\text{CD}(J, Q, \mu) := J \oplus s_0 J$, with multiplication and involution given by

$$\begin{aligned} (j_1 + s_0 j'_1)(j_2 + s_0 j'_2) &= j_1 j_2 + \mu(j'_1 j'_2)^{\theta} + s_0(j_1^\theta j'_2 + (j'_1)^\theta j_2^\theta), \\ \overline{(j + s_0 j')} &= j - s_0 j'^\theta, \end{aligned}$$

is a structurable algebra of skew-dimension one; see [AF1, Theorem 6.6].

We start with the central simple biquaternion algebra equipped with the reduced norm Nrd .

Lemma 7.10. *Let $Q_1 \otimes_K Q_2$ be a biquaternion algebra over a field K . Then the reduced norm Nrd is a Jordan norm of degree 4 of the Jordan algebra $(Q_1 \otimes_K Q_2)^+$.*

Proof. The central simple algebra $Q_1 \otimes_K Q_2$ has degree 4, so its reduced norm has degree 4 and the trace form is bilinear nondegenerate. Since the Jordan algebra arises from a biquaternion algebra, we have $U_j j' = j j' j$, and it follows that $\text{Nrd}(U_j j') = \text{Nrd}(j)^2 \text{Nrd}(j')$. \square

In order to apply the Cayley–Dickson construction to $(Q_1 \otimes Q_2)^+$, we have to determine the trace map T associated to Nrd explicitly. Since T is bilinear, it suffices to compute its value for elements of the form $a \otimes b$ in $Q_1 \otimes Q_2$.

It turns out that

$$T(a \otimes b, a' \otimes b') = \text{Trd}(a, a') \text{Trd}(b, b') \quad \forall a, a' \in Q_1, b, b' \in Q_2,$$

where Trd is the reduced trace. For $a = a_1 + a_2 \ell + a_3 m + a_4 \ell m \in Q_1$, $b = b_1 + b_2 n + b_3 r + b_4 nr \in Q_2$, we have that $T(a \otimes b, 1 \otimes 1) = 4a_1 b_1$, so

$$(a \otimes b)^\theta = -a \otimes b + 2a_1 b_1(1 \otimes 1).$$

From now on we assume $Q_1 = (-s_2, -s_3)_K$ and $Q_2 = (-s_{46}, -s_{56})_K$. Using the isomorphism ψ from Lemma 7.3 between $Q_1 \otimes_K Q_2$ and $X|_K$, θ acts on $X|_K$ as follows:

$$\left(t_{1_1} + \sum_{i \in \mathcal{I}} t_{i_1} v_i\right)^\theta = t_{1_1} - \sum_{i \in \mathcal{I}} t_{i_1} v_i \quad \text{for } t_{i_1} \in K. \quad (7.3)$$

Theorem 7.11. *The twisted Jordan biquaternion algebra X defined in Theorem 7.6 is isomorphic to $\text{CD}((Q_1 \otimes Q_2)^+, \text{Nrd}, \gamma^2)$.*

Proof. Applying the Cayley–Dickson process on $(Q_1 \otimes_K Q_2)^+ \cong X|_K$ we have $\text{CD}((Q_1 \otimes_K Q_2)^+, \text{Nrd}, \gamma^2) = X|_K \oplus s_0 X|_K$, where the multiplication and involution on $X|_K \oplus s_0 X|_K$ are as in Definition 7.9.

We now define a K -vector space isomorphism χ from X to $X|_K \oplus s_0 X|_K$:

$$\begin{aligned} (t_{1_1} + \gamma t_{1_2})v_1 + \sum_{i \in I} (t_{i_1} + \gamma t_{i_2})v_i \\ \mapsto (t_{1_1}v_1 + \sum_{i \in I} t_{i_1}v_i) + s_0(-t_{1_2}v_1 + \sum_{i \in I} t_{i_2}v_i) \end{aligned}$$

for all $t_{i_1}, t_{i_2} \in K$.

We first check that the involution from Theorem 7.6 is the same as the one we get from the Cayley–Dickson process.

$$\begin{aligned} \overline{\chi(t_{1_1}v_1 + \sum_{i \in I} t_{i_1}v_i)} &= \chi(t_{1_1}^\sigma v_1 + \sum_{i \in I} t_{i_1}v_i) \\ &= (t_{1_1}v_1 + \sum_{i \in I} t_{i_1}v_i) + s_0(t_{1_2}v_1 + \sum_{i \in I} t_{i_2}v_i) \\ &= (t_{1_1}v_1 + \sum_{i \in I} t_{i_1}v_i) - s_0(-t_{1_2}v_1 + \sum_{i \in I} t_{i_2}v_i)^\theta \quad \text{by (7.3)} \\ &= \overline{\chi(t_{1_1}v_1 + \sum_{i \in I} t_{i_1}v_i)}. \end{aligned}$$

It follows immediately from Definition 7.9 that $\chi(x)\chi(y) = \chi(x \star y)$ for all $x, y \in X|_K$. It requires a straightforward but lengthy calculation to verify that $\chi(x)\chi(y) = \chi(x \star y)$ for $x, y \in X$ as well. \square

Remark 7.12. In the E_6 -case, X is a twisted quaternion algebra; it is a structurable algebra isomorphic to $\text{CD}(Q_1^+, \text{Nrd}, \gamma^2)$. In the E_7 -case, X is a twisted quaternion algebra over a quadratic field extension; it is a structurable algebra isomorphic to $\text{CD}((Q_1 \otimes_K K(r))^+, \text{Nrd}, \gamma^2)$.

Remark 7.13. There is an alternative approach to Theorem 7.6 and Theorem 7.11, as follows. It is, in principle, possible to verify directly (with a very lengthy computation, or using Sage [Sage]) that

$$\chi(3x\pi(x)) = U_{\chi(x)}(s_0\chi(x))$$

for all $x \in X$, where the U in the right-hand side denotes the U -operator in the structurable algebra $\text{CD}((Q_1 \otimes Q_2)^+, \text{Nrd}, \gamma^2)$. It then follows immediately that X and $\text{CD}((Q_1 \otimes Q_2)^+, \text{Nrd}, \gamma^2)$ are isotopic, and since $\chi(v_1) = 1 \otimes 1$, it follows that these algebras are, in fact, isomorphic. It is then possible to compute the multiplication of $\text{CD}((Q_1 \otimes Q_2)^+, \text{Nrd}, \gamma^2)$ explicitly using this isomorphism χ .

7.3 The pseudo-quadratic spaces on E_6 and E_7

In this last section, we assume that $\Omega = (K, L, q, 1, X, \cdot, h, \theta)$ is a quadrangular algebra of type E_6 or E_7 . In this case, the rank one residues of the corresponding algebraic group (see Figures 12 and 13) are classical, and this fact is also visible at the level of the structurable algebras. Indeed, in [W2] it is shown that X can be made into a 4-dimensional (left) vector space over E or over the quaternion algebra $D = (E/K, s_2 s_3 s_4)$, respectively; moreover, there is, in both cases, an anisotropic pseudo-quadratic form on this vector space X , denoted by \hat{Q} , with the property that

$$x\pi(x) = \hat{Q}(x) * x \quad (7.4)$$

for all $x \in X$. (We have used the symbol $*$ to denote the scalar multiplication of X over E or D , respectively.) We refer to [W2, Definition 3.6, and Theorems 5.3 and 5.4] for more details.

It follows from equation (7.4) and Theorem 6.6 that the Freudenthal triple system corresponding to this pseudo-quadratic form as in Theorem 6.6 is similar to the Freudenthal triple system of X as a quadrangular algebra of type E_6 or E_7 . From Lemma 3.20 it follows that the corresponding structurable algebras are isotopic.

It is also interesting to note that it is shown in [W2, Theorem 5.12] that

$$q(\pi(x)) = N(\hat{Q}(x))$$

for all $x \in X$; both sides of this expression are, up to a constant, equal to the conjugate norm of the structurable algebra (see Theorem 5.4), where the left hand side corresponds to the structurable algebra arising from the quadrangular algebra of type E_6 or E_7 , and the right hand side corresponds to the structurable algebra arising from the pseudo-quadratic form.

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